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ON ORTHOGONAL EXPANSIONS OF THE SPACE OF VECTOR FUNCTIONS WHICH ARE SQUARE-SUMMABLE OVER A GIVEN DOMAIN AND THE VECTOR ANALYSIS OPERATORS

E. B. Bykhovskiy, N. V. Smirnov

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16. Abstract

The present paper is devoted to a study of the Hilbert space $L_2(\Omega)$ of vector functions.

The paper discusses a breakdown of $L_2(\Omega)$ into orthogonal subspaces, investigates the properties of the operators for projection onto these subspaces from the standpoint of preserving the differential properties of the vectors being projected, and examines in detail the properties of the operators.

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On Orthogonal Expansions of the Space of Vector Functions which are Square-Summable Over a Given Domain and the Vector Analysis Operators

E. B. Bykhovskiy, N. V. Smirnov

Introduction

The present paper is devoted to a study of the Hilbert space $L_2(\Omega)$ of vector functions: /5*

$$\mathbf{v}(x) = (v_1, v_2, v_3),$$

specified in the region Ω of a three-dimensional Euclidean space $x = (x_1, x_2, x_3)$. The scalar product in $L_2(\Omega)$ is defined by the equation:

$$(\mathbf{u}, \mathbf{v}) = \int_\Omega \sum_{k=1}^{3} u_k v_k dx.$$

The paper discusses a breakdown of $L_2(\Omega)$ into orthogonal subspaces, investigates the properties of the operators for projection onto these subspaces from the standpoint of preserving the differential properties of the vectors being projected, and examines in detail the properties of the operators rot [curl] and div.

The part that functional analysis and especially the theory of unlimited self-conjugate operators in Hilbert space have played in solving various problems of mathematical physics is well known.

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*Numbers in the margin indicate pagination in the foreign text.

In the majority of these problems the functional space $L_2(\Omega)$ plays the part of the fundamental Hilbert space. In view of this, problems of studying the various differential operators not in classes of continuously-differentiable functions, but rather so-called classes W_2^{ℓ}, i.e. classes of functions which can be quadratically added together with their derivatives up to a certain order l, occupy the primary attention. In fact this attempt to use general ideas and theorems of functional analysis to solve problems involving vector functions has necessitated a study of vector fields not from the classical standpoint, but rather that of Hilbert space $L_2(\Omega)$. The pioneering work in this direction is that of H. Weyl [4]. With a view to solving boundary-value problems for the Laplace operator, he broke down the space L_2 into three mutually orthogonal subspaces $\mathring{G}$, U and $\mathring{J}$ in the following manner. He defined $\mathring{G}$ as the closure in the norm of L_2 of the gradients of all continuously-differentiable scalar functions finite in Ω (equaling zero near boundary), $\mathring{J}$ as the closure of rot of all finite smooth vectors, and U as the orthogonal complement of $\mathring{G} \oplus \mathring{J}$ /6

The orthogonality of $\mathring{G}$ and $\mathring{J}$ is immediately verified. Thus, it is clear from the very structure of these subspaces that L_2 is the orthogonal sum of $\mathring{G}$, $\mathring{J}$ and U. One of the principal assertions of Weyl is that all vectors of U are gradients of harmonic functions. This is based on the lemma of harmonicity of the function in $L_2(\Omega)$ that is orthogonal to the Laplace operators of all finite functions. This lemma had been previously proved for the more general case of the polyharmonic operator of S. L. Sobolev [24].

On the other hand, it has long been known (cf. e.g. [10]) that any given smooth vector $\underline{u}(x)$ can be resolved into three orthogonal vectors $\underline{u}_1$, $\underline{u}_2$, $\underline{u}_3$, the first of which is grad$\phi(x)$ with $\phi(x)$ equaling zero at the boundary, $\underline{u}_3$ is a solenoidal

vector with normal element equaling zero and rot equaling rot $\underline{u}$, while $\underline{u}_2$ is grad of the haromnic function. Finding the vectors $\underline{u}_k$ for a given $\underline{u}$ involves solving boundary-value problems for the Laplace operator. It was perfectly clear that this resolution of $\underline{u}$ corresponds to the Weyl resolution. But, as mentioned above, Weyl devised his resolution for the purpose of solving those boundary-value problems which are encountered in determination of $\underline{u}_k$. However in the early 1950s other comparatively simple and general methods of O. A. Ladyzhenskaya (cf. [14], as well as [26]) for these problems had produced in a certain sense definitive results in spaces $W_2^{\ell+2}(\Omega)$. In particular, it was proved that the Laplace operator establishes a one-to-one correspondence between the classes $\overset{\circ}{W}{}_2^{\ell+2}$ and W_2^{ℓ}, where $\circ$ signifies that the functions in $W_2^{\ell+2}$ obey a certain of the conventional homogeneous boundary conditions. This naturally suggested the use of these findings for a more thorough investigation of the questions of orthogonal resolution of L_2. In particular it was important for various problems to establish whether projections of the vector $\underline{u}$ of $W_2(\Omega)$ will belong to $W_2^{\ell}(\Omega)$. As shown by O. A. Ladyzhenskaya for a finite domain and N. V. Smirnov for an unbounded domain, this is true if the boundary is regular. This has been proved by E. B. Bykhovskiy in [2] for the case of $W_2^1(\Omega)$ and projection onto $\overset{\circ}{J}$, as well as for projection onto $J = \overset{\circ}{J} + U$, when the vector has a null tangential component at the boundary.

Another group of issues arising from the use of the subspaces $\overset{\circ}{G}$, $\overset{\circ}{J}$ and U is as follows: smooth vectors of these subspaces have completely determined characteristic properties. For example[1], the plane vector of $\overset{\circ}{J}$ is the vector $\underline{u}$, for which $\operatorname{div}\underline{u} = 0$ and the normal component u_n is equal to zero on the boundary. We obtain $\overset{\circ}{J}$ by closure in the norm of L_2 of the set of all these smooth vectors. The question is in what sense

[1]For simplicity we shall now discuss a domain with boundary homeomorphic to a sphere.

this relation of the divergence and normal component equaling zero on the contour will be preserved for limit vectors $\underline{u}$ of $\overset{\circ}{J}$, generally not differentiable. One of the answers is that these vectors are orthogonal to all the gradient vectors or, which amounts to the same, they have a null "generalized divergence" and, in the "general sense", satisfy the boundary condition:

$$u_n|_s = 0.$$

But this answer is tautological. A different and more substantive answer was found: any given vector $\underline{u}$ of $\overset{\circ}{J}$ has a single valued representation in the form rot$\underline{w}$, where: /7

$$\mathbf{w} \in W_2^1(\Omega)$$

(i.e. is quadratically-summable over Ω together with its first derivatives), div$\underline{w}$ = 0 and:

$$w_n|_s = 0,$$

while $\|\mathbf{u}\|_{L_2(\Omega)}$ is equivalent to $\|\mathbf{w}\|_{W_2^1(\Omega)}$. This result, established by O. A. Ladyzhenskaya, was afterwards amplified by E. B. Bykhovskiy. Thus, he proved that any given vector $\underline{v}$ of

$$J(\Omega) = \overset{\circ}{J}(\Omega) \dotplus U(\Omega),$$

has a single-valued representation in this form, while vectors $\underline{u}$ of $\overset{\circ}{J}(\Omega)$ have a single-valued representation in the form:

$$\mathrm{rot}\,\mathbf{v}\ (\mathbf{v} \in W_2^1(\Omega),\ \mathrm{div}\,\mathbf{v} = 0$$

and the tangential component $\underline{v}$ on S is equal to zero), the norms

$$\|\mathbf{u}\|_{L_2} \text{ and } \|\mathbf{v}\|_{W_2^1}$$

being equivalent. Let us also note that these premises enabled E. B. Bykhovskiy to reduce boundary-value problems for the set of Maxwell equations to a solution of the Cauchy problem for an abstract equation of type:

$$\frac{dx}{dt} + iAx = f$$

with self-conjugate operator A, thereby solving these boundary value problems.

Let us finally discuss one more point which is clarified in the present paper, also arising (as the others above) in the study of problems of hydrodynamics of a viscous incompressible fluid. Let us presume that the domain Ω is multiple-connected. The Weyl resolution remains in force. However in the present case U contains gradients of multi-valued functions. It was necessary to explain the position of such gradients in L_2, for e.g. in hydrodynamic problems the operation of projection is used to find the pressure gradient of a function which, by definition of the problem, should be single-valued. It was found that if the set of all smooth solenoidal vectors with null normal component on the boundary (or, which amounts to the same, equaling zero near the boundary) is examined in L_2, it will take in all the gradients of multi-valued functions, and therefore such gradients will be absent from its orthogonal complement. This justifies the procedure given in [8] and [13] for solving stationary and nonstationary boundary-value problems for a viscous incompressible fluid in multiply-connected domains. The fact that the gradients of multi-valued functions and the solenoidal vectors that are not curls will basically lie in the subspace U was found by Weyl. But he did not describe in detail the properties of such vectors. Such a description was important, both for producing our results and for the applications. The most complete results in respect of resolutions for multiply-connected domains were obtained by

E. B. Bykhovskiy in [3]. They are explained in §1 of Chapter 3.

As is evident from the above, the studies of the resolution of L_2 undertaken after Weyl were basically motivated by the desire to solve hydrodynamical problems of a viscous incompressible fluid and electrodynamical problems, using for this the main achievements of functional analysis. The works of S. L. Sobolev [22,23] and S. G. Kreyn [3] belong to this same pursuit. In the works of S. L. Sobolev, in connection with the problems which they address, a resolution of L_2 is examined particularly in the case when Ω is all of 3-dimensional space. The work of Kreyn studies various field operators in the subspaces $\overset{\circ}{J}$ and J in connection with an investigation of the linearized stationary Navier-Stokes equations.

In §1 of Chapter I we discuss all of three-dimensional space E_3. From the known formula (1) giving the resolution of a finite vector into rot and grad of the vector and scalar potentials we derive the orthogonal resolution of $L_2(E_3)$ and the various properties of the operators for projection onto the subspaces. We examine the question of whether the vectors of $J(E_3)$ (the closure of rot of the vector potentials). In §2 of Chapter I we briefly expound the chief results of the paper of H. Weyl [4]. /8

In Chapter II we examine a bounded region homeomorphic to a sphere. We introduce others convenient for future definition of the subspaces of L_2 and prove they are equivalent to those of Weyl. We discuss the question of representing the vectors of $\overset{\circ}{J}$ and $J = J \oplus U$ in the form rot and the properties of the projection operators.

The same issues for a bounded multiply-connected domain are studied in §1 of Chapter III. In §2 of this chapter are additional considerations which enable some of the findings to

be extended to unbounded domains.

After this paper was sent to press we became aware of the work of K. Friedrichs "Differential Forms on Riemannian Manifolds" (Comm. Pure Appl. Math., VIII/2, November, 1955), which proves for the latter results that are similar to §2, Chapter II. Specifically, theorems 3.2 and 4.2, as well as the partial case of theorem 6.2 ($n = 1$), coincide with the findings of Friedrichs. Even so, they were obtained concurrently, independently, and using a different procedure. The question of the representability of any given vector of $\overset{\circ}{J}$ in the form of the curl of a solenoidal vector with null normal component on the boundary, as mentioned above, was decided by O. A. Ladyzhenskaya as far back as early 1954, reported at seminars, and afterwards used in works on hydrodynamics (cf., e.g., [9]), but unfortunately the result was not published in the explicit form. These investigations were continued by one of the authors, reporting on the findings at a meeting of the department at the end of 1955. In view of the work on their applications to the Maxwell equations, the paper [2], giving a significant amount of the findings of the present work, was sent to press in 1956.

One of the important differences between our procedure and that of Friedrichs consists in the method of deriving inequalities (16) and (22). Friedrichs extends the form onto the "duplicate" of the particular domain by an even or odd method, depending on the type of boundary conditions. This approach is able to avoid a treatment of the boundary and reduces the question to an evaluation of the Dirichlet integral over the interior subregion. But our treatment is done directly in the closed domain. This method has the advantage that it allows a similar treatment of the case when one condition is specified at one portion of the boundary, another at the other (for the investigations of the limit integrals in

lemmas 4.2 and 5.2 are local in nature), as well as an investigation of the properties of the projection operators on $\overset{\circ}{G}$, $\overset{\circ}{J}$ and U for vectors of $W_2^n(\Omega)$ when n > 1. Of interest is Weyl's and Friedrichs' formulation of whether the evaluation of the vector has an elementary proof in terms of the vector's curl in the norm of L_2 (inequalities (17) and (23)). We did this by merely using the elementary properties of the potential of the volume-masses and the known estimate of the Dirichlet integral of a harmonic function in terms of the Dirichlet integral along the boundary from its limit value. For a convex domain (in particular the polygon discussed by Friedrichs in the introduction of his article), the proof is even more simple.[1] /9

Let us also note that when investigating the existence of derivatives above the first order in a closed domain (cf. [2], theorems 6.2 and 5.3 and the remark after theorem 4.2) the method of continuation used by Friedrichs seems to us inapplicable (at any rate in its unchanged form), for the derivatives above the first order are not quadratically-summable for the continued forms.

Chapter I. The Space $L_2(E_3)$ and the Weyl Resolution for any Given Domain Ω

§1. The Space $L_2(E_3)$

Let $L_2(E_3)$[2] be a Hilbert space of real vector functions:

$$\mathbf{u}(x) = (u_1(x),\ u_2(x),\ u_3(x)),$$

quadratically-summable over all of 3-dimensional Euclidean space E_3[3], with scalar product:

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[1]Cf. the remark to lemmas 4.2 and 5.2

[2]Hereafter in this chapter we shall ordinarily omit the symbol E_3.

[3]All of the results are also valid for complex vectors.

$$(\mathbf{u}, \mathbf{v}) = \int_{E_3} \mathbf{u}(x) \cdot \mathbf{v}(x)\, dx. \quad ^1$$

It is known that smooth finite vectors (i.e. vanishing outside a finite region, peculiar to each vector) are dense in $L_2(E_3)$ and each such vector can be represented as [10]:

$$\mathbf{u}(x) = \frac{1}{4\pi} \operatorname{rot} \int_{E_3} \frac{\operatorname{rot} \mathbf{u}(y)}{|x-y|} dy - \frac{1}{4\pi} \operatorname{grad} \int_{E_3} \frac{\operatorname{div} \mathbf{u}(y)}{|x-y|} dy \equiv \operatorname{rot} \mathbf{A} + \operatorname{grad} p, \tag{1}$$

where the signs rot and grad stand before a smooth vector and smooth function that diminish at infinity as $\frac{1}{|x|^2}$ with derivatives diminishing as $\frac{1}{|x|^3}$.

Let us define two subspaces in the Hilbert space L_2. $G(E_3)$ is the closure in L_2 of the set of vectors of type $\operatorname{grad} p(x)$, where p are smooth functions diminishing at infinity as $\frac{1}{|x|^2}$ with first derivatives diminishing as $\frac{1}{|x|^3}$. $J(E_3)$ is the closure in L_2 of the set of vectors of type $\operatorname{rot} \underline{A}$, the order of diminution at infinity of $\underline{A}$ and its derivatives being the same as for p.

The sets that are closed are orthogonal in L_2, for:

$$\int_{E_3} \operatorname{grad} p \cdot \operatorname{rot} \mathbf{A}\, dx = \int_{E_3} p \operatorname{div} \operatorname{rot} \mathbf{A}\, dx = 0.$$

Thus the subspaces G and J are also orthogonal.

Formula (1) gives the resolution of a finite $\underline{u}$ into components in J and G. It generalizes at once to the case of any given $\mathbf{u} \in L_2$.

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[1]The dot signifies the scalar product in the sense of vector algebra.

Theorem 1.1.

$$\text{a) } L_2 = G \ominus J. \qquad (2)$$

b) If

$$\mathbf{u} \in W_2^n(E_3),$$

where W_2^n is the space of S. L. Sobolev [21], its projections on G and J will also belong to $W_2^n(E_3)$, their norms in W_2^n not exceeding $\|\mathbf{u}\|_{W_2^n}$. /10

Proof. a) (2) immediately follows from (1), for any given vector $\mathbf{u} \in L_2$ is the limit of finite vectors $\underline{u}_n$ with respect to the norm of L_2. By virtue of the mutual orthogonality of the vectors rot$\underline{A}_n$ and gradp, corresponding to these $\underline{u}_n$ by formula (1), the former converge in L_2. Passing to the limit in (1), which was written for $\underline{u}_n$, we obtain for $\underline{u}$ the resolution:

$$\mathbf{u} = \mathbf{u}_J \oplus \mathbf{u}_G, \text{ где } \mathbf{u}_J \in J; \ \mathbf{u}_G \in G,$$

which in fact proves the theorem. b) If $\underline{u}$ is a smooth finite vector, the inequality:

$$\|\mathbf{u}_J\|_{W_2^n(E_3)} \leqslant \|\mathbf{u}\|_{W_2^n(E_3)} \qquad (*)$$

and an analogous one for $\underline{u}_G$ follow from the fact that resolution (1) can be written for any given derivative D of the vector $\underline{u}$:

$$D\mathbf{u} = D \operatorname{rot} \mathbf{A} + D \operatorname{grad} p,$$

while $D \operatorname{rot} \mathbf{A} = \operatorname{rot} D\mathbf{A} \in J$; $D \operatorname{grad} p = \operatorname{grad} Dp \in G$.

For any given vector

$$\mathbf{u} \in W_2^n(E_3)$$

a confirmation of the theorem follows from the fact that, as is known, it can be approximated by smooth finite vectors $\underline{u}_n$ with respect to the norm of $W_2^n(E_3)$.

By virtue of inequality (*) for $\underline{u}_J$ and $\underline{u}_G$, the vectors $\underline{u}_{nJ}$ and $\underline{u}_{nG}$ converge in $W_2^n(E_3)$, and therefore their limits - the vectors $\underline{u}_J$ and $\underline{u}_G$ - also belong to this space, the evaluation of

$$\|\mathbf{u}_J\|_{W_2^n} \quad \text{and} \quad \|\mathbf{u}_G\|_{W_2^n}$$

in terms of $\|\mathbf{u}\|_{W_2^n}$ being conserved.

Theorem 2.1. The gradients of smooth finite functions p are dense in G, while the curls of smooth finite vectors $\underline{A}$ are dense in J.

This statement becomes obvious if we consider that p and A, used to determine G and J, belong to $W_2^1(E_3)$ and therefore can be approximated by finite p_n and $\underline{A}_n$ with respect to the norm of $W_2^1(E_3)$.

Let us discuss the question of the representation of any given vector of J in the form of the curl of another vector. If such a representation is possible then e.g. for a vector of J that diminishes as $\frac{1}{|x|^2}$ at infinity, the vector of which it is the curl should diminish as $\frac{1}{|x|}$, i.e. it does not belong to $L_2(E_3)$. Consequently, an outlet from $L_2(E_3)$ is necessary in this matter. The space D_1 is introduced in this connection, being the closure of plane finite functions ψ in the norm of

$$\|\psi\|_1^2 = = \int_{E_3} \sum_{k=1}^{3} (\psi_{x_k})^2 \, dx \text{ [13].}$$

We shall denote a similar space for vectors $\underline{\psi}$ as $\underline{D}_1$. Let us

explain several of the properties of the elements of D_1. For this purpose we shall demonstrate that, for any smooth finite function ψ, the inequality is valid [12,13]:

$$\int_{E_3} \frac{\psi^2(y)}{|x-y|^2}\, dy \leqslant 4\|\psi\|_1^2. \tag{3}$$

We have: /1

$$\int_{E_3} \frac{\psi^2(y)}{|x-y|^2}\, dy = \int_{E_3} \psi^2(y) \sum_{k=1}^{3} \frac{\partial}{\partial y_k} \frac{x_k-y_k}{|x-y|^2}\, dy =$$

$$= -2 \int_{E_3} \sum_{k=1}^{3} \psi\psi_{y_k} \frac{x_k-y_k}{|x-y|^2}\, dy \leqslant 2 \left(\int_{E_3} \frac{\psi^2}{|x-y|^2} dy \right)^{1/2} \left(\int_{E_3} \sum_{k=1}^{3} \psi_{y_k}^2 dy \right)^{1/2},$$

from which (3) also follows.

It follows from (3) that functions in D_1 are locally quadratic-summable, while their derivatives belong to $L_2(E_3)$. The same is also true of vectors in $\underline{D}_1$.

Theorem 3.1. Any vector $\underline{v}$ of J has a single-valued representation as:

$$\mathbf{v} = \operatorname{rot} \mathbf{A}, \quad \text{where } \mathbf{A} \in \mathbf{D}_1,$$

while:

$$\|\mathbf{A}\|_1 = \|\mathbf{v}\|_{L_2}. \tag{4}$$

Any vector of G has a single-valued representation as gradp, where:

$$p \in D_1 \text{ and } \|p\|_{D_1} = \|\operatorname{grad} p\|_{L_2}.$$

Proof. Let $\underline{v}$ at first be a smooth finite solenoidal vector. Then:

$$\mathbf{v} = \frac{1}{4\pi}\,\mathrm{rot}\,\mathrm{rot}\int_{E_3} \frac{\mathbf{v}(y)}{|x-y|}\,dy,$$

and it is sufficient to posit that:

$$\mathbf{A} = \frac{1}{4\pi}\,\mathrm{rot}\int_{E_3} \frac{\mathbf{v}(y)}{|x-y|}\,dy. \tag{5}$$

The finite nature of $\underline{v}$ and formula (5) lead to the estimates:

$$|\mathbf{A}| \leqslant \frac{C(\mathbf{A})}{|x|^2}\,; \quad \left|\frac{\partial \mathbf{A}}{\partial x_k}\right| \leqslant \frac{C(\mathbf{A})}{|x|^3}, \tag{6}$$

from which it follows that:

$$\mathbf{A} \in W_2^1(E_3)$$

and moreover:

$$\mathbf{A} \in \mathbf{D}_1\ (4)$$

follows directly from the fact that, in view of the solenoidal character of $\underline{A}$ and the estimates (6), we have:

$$\int_{E_3} (\mathrm{rot}\,\mathbf{A})^2\,dx = \int_{E_3} \sum_{i,k=1}^{3} \left(\frac{\partial A_i}{\partial x_k}\right)^2 dx.$$

Now let $\underline{v}$ be any vector in J. We shall approximate it with respect to norm of L_2 by finite solenoidal vectors $\underline{v}_n$ (which is possible from theorem 2.1). In virtue of (4), the respective $\underline{A}_n$ converge in the norm of $\underline{D}_1$ to a certain:

$$\mathbf{A} \in \mathbf{D}_1,$$

while $\underline{v}$ = rot$\underline{A}$. Obviously (4) is conserved. The uniqueness of the representation $\underline{v}$ = rot$\underline{A}$ follows immediately from (4).

The statement of the theorem that concerns vectors of G follows immediately from the definition of D_1 and the denseness in G of gradients of finite vectors.

§2. The Orthogonal Resolution of Weyl

In this section we explain the chief findings of H. Weyl /12
[4]. Let Ω be any open set of 3-dimensional space, E_3 and $L_2(\Omega)$[1] a Hilbert space of 3-dimensional vectors quadratically-summable over Ω, with scalar product:

$$(\mathbf{u},\ \mathbf{v})=\int \mathbf{u}\cdot\mathbf{v}\,dx.^{*} \qquad 1$$

the vectors and functions that equal zero in the neighborhood of the boundary and infinity (if Ω is an unbounded set) we shall call finite.

Let us examine in L_2 the subspaces:

$\mathring{G}$ - the closure in L_2 of vectors gradϕ, where ϕ is a finite function,

$\mathring{J}$ - the closure in L_2 of the vectors rot$\underline{v}$, where $\underline{v}$ is a finite vector.

The orthogonality of $\mathring{G}$ and $\mathring{J}$ is obvious.

Let G be the orthogonal complement of $\mathring{J}$, J the orthogonal complement of $\mathring{G}$, and U the intersection of G and J.

The resolutions:

$$\begin{matrix} G=\mathring{G}\oplus U \\ J=\mathring{J}\oplus U \end{matrix};\quad L_2=G\oplus\mathring{J}=J\oplus\mathring{G}=\mathring{G}\oplus U\oplus\mathring{J}. \qquad (7)$$

follow directly from the definition of these subspaces.

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[1]We afterwards omit the symbol Ω.

Theorem 4.1. If $f \in U$, it is an infinitely-differentiable harmonic vector:

$$\Delta \mathbf{f} = 0, \operatorname{rot} \mathbf{f} = 0 \; u \; \operatorname{div} \mathbf{f} = 0.$$

In the proof we shall avail ourselves of the following lemma, obtained by S. L. Sobolev [24] and independently by H. Weyl [4].

If:

$$\int \eta \Delta \xi dx = 0,$$

where ξ is any smooth finite function, then η is a harmonic function.

Let $f \in U$, i.e.

$$\int \mathbf{f} \cdot \operatorname{rot} \mathbf{v} dx = 0; \int \mathbf{f} \cdot \operatorname{grad} \varphi dx = 0 \tag{8}$$

for any smooth finite $\underline{v}$ and ϕ.

Let us assume that:

$$\mathbf{v} = \operatorname{rot} \mathbf{w} \text{ and } \varphi = \operatorname{div} \mathbf{w},$$

where $\underline{w}$ is finite. Then, using the identity:

$$\Delta \mathbf{w} = \operatorname{grad} \operatorname{div} \mathbf{w} - \operatorname{rot} \operatorname{rot} \mathbf{w},$$

we obtain:

$$\int \mathbf{f} \Delta \mathbf{w} dx = 0,$$

whence we conclude that the components of $\underline{f}$ are harmonic functions.

We have:

$$\left.\begin{aligned} \operatorname{div} \varphi \mathbf{f} &= \mathbf{f} \operatorname{grad} \varphi + \varphi \operatorname{div} \mathbf{f}, \\ \operatorname{div} \mathbf{f} \times \mathbf{v} &= \mathbf{v} \cdot \operatorname{rot} \mathbf{f} - \mathbf{f} \cdot \operatorname{rot} \mathbf{v}. \end{aligned}\right\} \tag{9}$$

Substituting $\underline{f}\,\mathrm{grad}\phi$ from (9) into (8) and integrating with respect to Ω, we have (using the fact that $\int\limits_{\Omega} \mathrm{div}\, \varphi f dx = 0$):

$$\int\limits_{\Omega} \varphi \,\mathrm{div}\, f dx = 0$$

and, since ϕ is arbitrary, $\mathrm{div}\underline{f} = 0$. Similarly, we establish that $\mathrm{rot}\underline{f} = 0$. It follows from this that $\underline{f} = \mathrm{grad}h$, where $h(x)$ is a function harmonic in Ω, which may be multi-valued. Q.E.D. /13

A considerable part of Weyl's work is devoted to a treatment of multiply-connected regions. He obtains an estimate for the unidimensional periods of potential vectors and 2-dimensional periods of solenoidal vectors in terms of the norms of these vectors in L_2. As a consequence it is found that the periods of these vectors are equal in magnitude to the periods of their projections onto harmonic subspace.

As mentioned in the introduction, a detailed knowledge of the structure of subspace U on the basis of the well-known properties of boundary-value problems was important for us. Therefore we shall forego an explanation of this part of Weyl's paper.

Chapter II. A Bounded Domain with Boundary Homeomorphic to a Sphere

§1. Definition of the Subspaces of Weyl by Means of Boundary-Value Problems

Let Ω be a bounded domain in the space E_3 with boundary S homeomorphic to a sphere and rather smooth.[1]

We introduce a series of subspaces, retaining the desig-

[1] We shall explain the requisite smoothness S in various concrete instances.

nations of Chapter I, for afterwards we shall establish that they coincide with the subspaces of Weyl therein employed.

Let:

$\overset{\circ}{G}$ - the closure in $L_2(\Omega)$ of the lineal $\tilde{\overset{\circ}{G}}$ of gradients of smooth functions ϕ that vanish on S.

$\overset{\circ}{J}$ - the closure in $L_2(\Omega)$ of the lineal $\tilde{\overset{\circ}{J}}$ of smooth vectors $\underline{v}$ for which:

$$\operatorname{div} \mathbf{v}=0, \ v_n|_S=0$$

(v_n is the normal component of $\underline{v}$).

J - the closure in $L_2(\Omega)$ of the lineal $\tilde{J}$ of smooth vectors $\underline{v}$ for which div$\underline{v}$ = 0.

G - the closure in $L_2(\Omega)$ of the gradients of all functions that are smooth in $\bar{\Omega}$

U - the closure in $L_2(\Omega)$ of the lineal $\tilde{U}$ of gradients of harmonic functions h(x) that are continuously differentiable in $\bar{\Omega}$.

It is easy to see that the lineals $\tilde{\overset{\circ}{G}}, \tilde{U}$ and $\tilde{\overset{\circ}{J}}$ are orthogonal in pairs and that $\tilde{G}$ is orthogonal to $\tilde{\overset{\circ}{J}}$, while $\tilde{G}$ is orthogonal to $\tilde{J}$.

<u>Theorem 1.2</u>. Any smooth vector $\underline{u}$ using the Newton potential and the solution of boundary-value problems for the Laplace operator can be represented as the sum:

$$\mathbf{u}=\mathbf{u}_1+\mathbf{u}_2+\mathbf{u}_3, \tag{10}$$

where:

$$\mathbf{u}_1 \in \tilde{\overset{\circ}{G}}; \ \mathbf{u}_2+\mathbf{u}_3 \in \tilde{J}; \ \mathbf{u}_2 \in \tilde{U}; \ \mathbf{u}_3 \in \tilde{\overset{\circ}{J}}, \ \mathbf{u}_1+\mathbf{u}_2 \in \tilde{G}.$$

Consequently,

$$L_2 = \mathring{G} \dotplus U \oplus J = \mathring{G} \oplus J = G \oplus \mathring{J}.^{**}$$ [1] (11)

Proof. It is sufficient to set $\underline{u}_1 = \text{grad}\phi$, where ϕ is the solution of the problem: /14

$$\Delta\varphi = \text{div}\,\mathbf{u};\ \varphi|_s = 0,$$

for then:

$$\mathbf{u}_1 \in G, \text{ while } \mathbf{u} - \mathbf{u}_1 \in \tilde{J}$$

in view of:

$$\text{div}(\mathbf{u} - \mathbf{u}_1) = \text{div}\,\mathbf{u} - \text{div}\,\text{grad}\,\varphi = \text{div}\,\mathbf{u} - \Delta\varphi = 0.$$

To obtain $\underline{u}_3$ let us examine the problem:

$$\left.\begin{aligned} \text{rot}\,\mathbf{u}_3 &= \text{rot}\,\mathbf{u}, \\ \text{div}\,\mathbf{u}_3 &= 0, \\ u_{3n}|_s &= 0. \end{aligned}\right\} \tag{12}$$

In fact, by virtue of the second and third condition of this problem, its solution is:

$$\mathbf{u}_3 \in \mathring{J}.$$

And as for the difference $\underline{u}-\underline{u}_3$, it is orthogonal to $\mathring{J}$. . Actually, since:

$$\text{rot}(\mathbf{u} - \mathbf{u}_3) = 0, \text{ then } \mathbf{u} - \mathbf{u}_3 = \text{grad}\,\varphi,$$

and if $\underline{v}$ is any vector of J, then:

$$\int_\Omega (\mathbf{u} - \mathbf{u}_3) \cdot \mathbf{v}\,dx = \int_\Omega \text{grad}\,\varphi \cdot \mathbf{v}\,dx = \int_\Omega \varphi\,\text{div}\,\mathbf{v}\,dx + \int_S \varphi v_n\,dS = 0.$$

[1] If the introduced subspaces are established to coincide with those of Weyl (Theorem 5.2), these resolutions coincide with those of (7).

In solving problem (12) it would be sufficient to refer e.g. to [2] or to the lemma 2.2 given below. But for the ultimate estimates (e.g. for theorem 6.2) it is more convenient to solve it somewhat differently. If we do not heed the condition:

$$u_{3n}|_S = 0,$$

the sought vector $\underline{u}_3^1$ can be obtained as follows: let us extend $\underline{u}$ beyond Ω, preserving the smoothness, so that outside a certain fixed region $\Omega_1 \supset \Omega$, $\underline{u} = 0$ holds. We can assume that:

$$\mathbf{u}_3^1 = \operatorname{rot} \frac{1}{4\pi} \int_{\Omega_1} \frac{\operatorname{rot} \mathbf{u}(y)}{|x-y|} dy.$$

Actually, if:

$$\mathbf{v} = \frac{1}{4\pi} \int_{\Omega_1} \frac{\operatorname{rot} \mathbf{u}(y)}{|x-y|} dy,$$

then:

$$\operatorname{rot} \mathbf{u}_3^1 = \operatorname{rot} \operatorname{rot} \mathbf{v} = \operatorname{grad} \operatorname{div} \mathbf{v} - \Delta \mathbf{v} = \operatorname{rot} \mathbf{u},$$

for:

$$\operatorname{div}_x \mathbf{v} = \int_{\Omega_1} \frac{\operatorname{div}_y \operatorname{rot} \mathbf{u}(y)}{|x-y|} dy = 0.$$

In order for the sought vector $\underline{u}_3$ to satisfy the condition:

$$u_{3n}|_S = 0,$$

it is sufficient to set:

$$\mathbf{u}_3 = \mathbf{u}_3^1 + \operatorname{grad} \psi,$$

where ψ is the solution of the Neuman problem in Ω:

$$\Delta \psi = 0; \quad \frac{\partial \psi}{\partial n}\Big|_S = -u_{3n}^1|_S.$$

Thus, $\underline{u}_3$ is found.

As for the vector:

$$\mathbf{u}_2 = \mathbf{u} - \mathbf{u}_1 - \mathbf{u}_3,$$

the fact that it belongs to $\tilde{U}$ follows from:

$$\operatorname{rot} \mathbf{u}_2 = \operatorname{rot}(\mathbf{u} - \mathbf{u}_3) - \operatorname{rot} \mathbf{u}_1 = 0,$$

while:

$$\operatorname{div} \mathbf{u}_2 = \operatorname{div}(\mathbf{u} - \underline{\mathbf{u}}_1) - \operatorname{div} \mathbf{u}_3 = 0,$$

so that $\underline{u}_2$ = gradh, where h is a harmonic function, smooth in $\bar{\Omega}$.

Since $\mathbf{u}_1 + \mathbf{u}_2 = \operatorname{grad}(\varphi + h)$, it follows that $\mathbf{u}_1 + \mathbf{u}_2 \in \tilde{G}$.

The orthogonal resolutions (11) follow immediately from the definition of all the introduced subspaces and the proven representation of smooth vectors.

A useful consequence for the solution of physical problems[1] and studying the operator rot emerges from this theorem.

<u>Theorem 2.2</u>. Smooth solenoidal vectors with null tangential component on the boundary are dense in J.

In fact, if we regard $\underline{u}$ in (10) as finite, then $\underline{u}-\underline{u}_1$, which is its projection onto J, has:

$$(u - u_1)_\tau |_S = u_\tau |_S - (\operatorname{grad} \varphi)_\tau |_S = 0$$

(the index τ designates the tangential component).

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[1]E.g., the initial boundary-value problem for the set of Maxwell equations with an ideally conducting boundary.

Since finite $\underline{u}$ are dense in L_2, their projections are dense in J, Q.E.D.

§2. Representation of Vectors of Subspaces J and $\overset{\circ}{J}$ in the Form of Curls

I. Representation of Smooth Vectors of J and $\overset{\circ}{J}$ in the Form of Curls

Lemma 1.2. Let $\mathbf{u} \in \overset{\circ}{J}$, , i.e. $\operatorname{div} \mathbf{u} = 0,\ u_n|_S = 0.$
Then:

$$\mathbf{u} = \operatorname{rot} \mathbf{v}, \tag{13}$$

where:

$$\operatorname{div} \mathbf{v} = 0;\ v_\tau|_S = 0.$$

This is an unique representation. /1

The uniqueness of the vector $\underline{v}$ emerges from the fact that, if rot$\underline{v}$ = 0 and div$\underline{v}$ = 0, then $\underline{v}$ = gradh, where h is a harmonic function. From the condition:

$$v_\tau|_S = (\operatorname{grad} h)_\tau|_S = 0$$

there follows the constancy of h on S and, consequently, in Ω as well, i.e. $\underline{v}$ equals zero. We shall now prove the existence of representation (13). We construct:

$$\mathbf{v}_{\mathrm{I}} = \frac{1}{4\pi} \operatorname{rot} \int_\Omega \frac{\mathbf{u}(y)}{|x-y|} dy;$$

we have div$\underline{v}_{\mathrm{I}}$ = 0. Let us determine rot$\underline{v}_{\mathrm{I}}$:

$$\operatorname{rot} \mathbf{v}_{\mathrm{I}} = \frac{1}{4\pi} \operatorname{grad} \operatorname{div} \int_\Omega \frac{\mathbf{u}(y)}{|x-y|} dy - \frac{1}{4\pi} \Delta \int_\Omega \frac{\mathbf{u}(y)}{|x-y|} dy,$$

but

$$\operatorname{div} \int_\Omega \frac{\mathbf{u}(y)}{|x-y|} dy = \int_\Omega \frac{\operatorname{div} \mathbf{u}(y)}{|x-y|} dy - \int_S \frac{u_n}{|x-y|} dS_y = 0.$$

and therefore $\operatorname{rot} \mathbf{v}_I = \mathbf{u}$.

Let ℓ be any closed contour on S that includes a piece of the surface S^1.

Then:

$$\int_l \mathbf{v}_{I\tau} dl = \int_l \mathbf{v}_I \cdot d\mathbf{l} = \int_{S^1} (\operatorname{rot} \mathbf{v}_I)_n dS = \int_{S^1} u_n dS = 0.$$

Consequently $\mathbf{v}_{I\tau}|_S$ is the gradient of a certain function φ, which is determined on S.

It is now sufficient to assume that:

$$\mathbf{v} = \mathbf{v}_I + \mathbf{v}_{II}, \text{ where } \mathbf{v}_{II} = \operatorname{grad} \varphi,$$

while ϕ is the solution of the problem $\Delta\phi = 0$ in Ω and $\varphi|_S = -\varphi_0$.

Lemma 2.2.[1] Let $\mathbf{w} \in \bar{J}$, i.e. $\underline{w}$ is smooth and $\operatorname{div}\underline{w} = 0$. Then $\underline{w} = \operatorname{rot}\underline{u}$, where $\mathbf{u} \in \bar{J}$. This is an unique representation.

The uniqueness follows from the fact that, if $\operatorname{rot}\underline{u} = 0$ and $\mathbf{u} \in \bar{J}$, then $\underline{u} = \operatorname{grad} h$, where h is a harmonic function and:

$$\left.\frac{\partial h}{\partial n}\right|_S = 0,$$

i.e. h is constant in Ω and $\underline{u} = 0$.

Let us denote by Ω_1 any region that contains the region Ω, and by S_1 its boundary, which we may regard as smooth. Let ψ be the solution of the Neuman problem in the region $\Omega_1 - \Omega$:

$$\Delta\psi = 0; \quad \left.\frac{\partial \psi}{\partial n}\right|_S = -w_n|_S; \quad \left.\frac{\partial \psi}{\partial n}\right|_{S_1} = 0.$$

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[1] Cf. the book of N. Ye. Kochin [10].

The vector:

$$\mathbf{k} = \int_{\Omega} \frac{\mathbf{w}(y)}{|x-y|}\, dy + \int_{\Omega_1-\Omega} \frac{\operatorname{grad}\psi(y)}{|x-y|}\, dy$$

is solenoidal in Ω. In fact:

$$\operatorname{div}\left[\int_{\Omega} \frac{\mathbf{w}(y)}{|x-y|}\, dy + \int_{\Omega_1-\Omega} \frac{\operatorname{grad}\psi(y)}{|x-y|}\, dy\right] = \int_{\Omega} \frac{\operatorname{div}\mathbf{w}(y)}{|x-y|}\, dy -$$

$$-\int_{S} \frac{w_n}{|x-y|}\, dS_y + \int_{\Omega_1-\Omega} \frac{\operatorname{div}\operatorname{grad}\psi(y)}{|x-y|}\, dy - \int_{S+S_1} \frac{\operatorname{grad}_n\psi(y)}{|x-y|}\, dS_y =$$

$$= -\int_{S} \frac{w_n + \frac{\partial\psi}{\partial n}}{|x-y|}\, dS_y = 0.$$

Let us examine in Ω the solenoidal vector:

$$\mathbf{u}_{I} = \frac{1}{4\pi} \operatorname{rot}\mathbf{k}.$$

It is easy to see that $\operatorname{rot}\underline{u}_I = \underline{w}$. In fact, exploiting the solenoidal nature of $\underline{k}$, we have:

$$\operatorname{rot}\mathbf{u}_{I} = -\frac{1}{4\pi}\Delta\mathbf{k} = -\frac{1}{4\pi}\Delta\int_{\Omega} \frac{\mathbf{w}(y)}{|x-y|}\, dy - \frac{1}{4\pi}\Delta\int_{\Omega_1-\Omega} \frac{\operatorname{grad}\psi(y)}{|x-y|}\, dy = \mathbf{w}.$$

It is now sufficient to set:

$$\mathbf{u} = \mathbf{u}_{I} + \mathbf{u}_{II}, \quad \text{where} \quad \mathbf{u}_{II} = \operatorname{grad}\varphi,$$

while φ is the solution of the problem $\Delta\phi = 0$ in Ω; $\left.\frac{\partial\varphi}{\partial n}\right|_S = -\left.u_{In}\right|_S$.

II. Estimates of Smooth Solenoidal Vectors with Various Boundary Conditions and of Their Derivatives in Terms of Their rot

Lemma 3.2. For any continuously-differentiable solenoidal vector $\underline{w}$ in $\bar{\Omega}$ the identity is valid:

$$\int_\Omega (\operatorname{rot} \mathbf{w})^2\, dx = \int_\Omega \sum_{i,\,k=1}^{3} \left(\frac{\partial w_i}{\partial x_k}\right)^2 dx + \int_S \left\{\mathbf{w} \times \operatorname{rot} \mathbf{w} \cdot \mathbf{n} - \frac{1}{2} \operatorname{grad} \mathbf{w}^2 \cdot \mathbf{n}\right\} dS. \tag{14}$$

Here w_i are the components of $\underline{w}$, $\underline{n}$ is the unit vector of the external normal to S.

For the proof, we shall convert the surface integral on the right into a volume integral.

We have:

$$\int_S \left\{\mathbf{w} \times \operatorname{rot} \mathbf{w} \cdot \mathbf{n} - \frac{1}{2} \operatorname{grad} \mathbf{w}^2 \cdot \mathbf{n}\right\} dS = \int_\Omega \left\{\operatorname{div} (\mathbf{w} \times \operatorname{rot} \mathbf{w}) - \frac{1}{2} \operatorname{div} \operatorname{grad} \mathbf{w}^2\right\} dx. \tag{15}$$

Using the formula for div of a vector product we have: /1

$$\operatorname{div} (\mathbf{w} \times \operatorname{rot} \mathbf{w}) = (\operatorname{rot} \mathbf{w})^2 - \mathbf{w} \cdot \operatorname{rot} \operatorname{rot} \mathbf{w} = (\operatorname{rot} \mathbf{w})^2 + \mathbf{w} \cdot \Delta \mathbf{w}$$

(we have made use of the fact that, in view of div$\underline{w}$ = 0, $\operatorname{rot} \operatorname{rot} \mathbf{w} = -\Delta \mathbf{w}$).

Further:

$$\frac{1}{2} \operatorname{div} \operatorname{grad} \mathbf{w}^2 = \frac{1}{2} \Delta (\mathbf{w}^2) = \sum_{i,\,k=1}^{3} \left(\frac{\partial w_i}{\partial x_k}\right)^2 + \mathbf{w} \cdot \Delta \mathbf{w}.$$

Substituting all of this into the right side of (15) we in fact obtain (14).

<u>Lemma 4.2.</u>[1] For smooth vectors $\underline{v}$ in $\bar{\Omega}$, which have div$\underline{v}$ = 0, $v_\tau|_S = 0$, the inequality applies:

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[1]Hereafter by C we shall designate various constants depending solely on Ω. We shall omit the symbols Ω in the designations $W_2^1(\Omega)$ and $L_2(\Omega)$.

$$\|\mathbf{v}\|_{H_2^1(\Omega)} \leqslant C\|\operatorname{rot} \mathbf{v}\|_{L_2(\Omega)} \tag{16}$$

(S is presumed to be twice continuously-differentiable).

First of all we establish that:

$$\int_\Omega \mathbf{v}^2 dx \leqslant C \int_\Omega (\operatorname{rot} \mathbf{v})^2 dx. \tag{17}$$

Let us assume that rot$\underline{v}$ = $\underline{u}$.

It is easy to see that then $\underline{u}$ satisfies the conditions of lemma 1.2, and consequently $\underline{v} = \underline{v}_I + \underline{v}_{II}$, where $\underline{v}_I$ and $\underline{v}_{II}$ are as in lemma 1.2. It is obvious that:

$$|\mathbf{v}_I| \leqslant C \int_\Omega \frac{|\mathbf{u}(y)|}{|x-y|^2} dy,$$

whence by virtue of the properties of potential-type integrals (cf. [21]):

$$\int_\Omega \mathbf{v}_I^2 dx \leqslant C \int_\Omega \mathbf{u}^2 dx \quad \text{и} \quad \int_S v_{I\tau}^2 ds \leqslant C \int_\Omega \mathbf{u}^2 dx. \tag{18}$$

For harmonic functions ϕ the inequality is valid:

$$\int_\Omega (\operatorname{grad} \varphi)^2 dx \leqslant C \int_S (\operatorname{grad} \varphi)_\tau^2 dS,$$

whence there follows:

$$\int_\Omega \mathbf{v}_{II}^2 dx \leqslant C \int_S v_{I\tau}^2 dS \leqslant C \int_\Omega \mathbf{u}^2 dx. \tag{19}$$

An even stronger inequality, containing in addition on the left:

$$\int_S \left(\frac{\partial \varphi}{\partial n}\right)^2 dS,$$

emerges for a singly-connected region from the works of Vishik

[5] and [6]. For a multiply-connected region this is proved by Eydus in [25]. We point out that:

$$\int_S \varphi^2 dS.$$

is also present on the right in [25], although it can be evaluated in elementary manner in terms of:

$$\int_S (\mathrm{grad}\,\varphi)^2 dS. \qquad 1$$

It is very probable that the weaker inequality which we require had also turned up in the past. From (18) and (19) there follows (17).

Let us now evaluate the derivatives of v. We shall apply the identity (14) to v and show that, thanks to the condition $v_\xi|_S=0$ the integral over S does not in fact contain derivatives of v. /1

We introduce the definitions:

(ξ_1, ξ_2, ξ_3) - local coordinates at the point 0 on S.
$(\tilde\xi_1, \tilde\xi_2, \tilde\xi_3)$ - local coordinates at point 0 (the axes ξ_3 and $\tilde\xi_3$ are normal to S).
v_k^1 - components of v in coordinates (ξ_1, ξ_2, ξ_3).
$\tilde v_k$ - components of v in coordinates $(\tilde\xi_1, \tilde\xi_2, \tilde\xi_3)$.

Let $\xi_3 = F(\xi_1,\xi_2)$ be the equation of S in the neighborhood of 0.

We have:

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[1] Using the fact that S is a surface without edge. The methods of the cited works are entirely elementary.

$$\mathbf{v} \times \operatorname{rot} \mathbf{v} \cdot \mathbf{n} + \frac{1}{2} \operatorname{grad} \mathbf{v}^2 \cdot \mathbf{n} = \frac{1}{2} \frac{\partial}{\partial \xi_3} \left(v_1^{1^2} + v_2^{1^2} + v_3^{1^2} \right) =$$
$$= \frac{1}{2} \frac{\partial}{\partial \xi_3} v_3^{1^2} = v_3^1 \frac{\partial v_3^1}{\partial \xi_3} = - v_3^1 \left(\frac{\partial v_2^1}{\partial \xi_2} + \frac{\partial v_1^1}{\partial \xi_1} \right),$$

but:

$$v_2^1 = \tilde{v}_3 \frac{-F_{\xi_2}}{\sqrt{F_{\xi_1}^2 + F_{\xi_2}^2 + 1}},$$

from which:

$$\left. \frac{\partial v_2^1}{\partial \xi_2} \right|_{\xi_1 = \xi_2 = 0} = (-v_2^1 F_{\xi_2 \xi_2})_{\xi_1 = \xi_2 = 0}$$

and similarly:

$$\left. \frac{\partial v_1^1}{\partial \xi_1} \right|_{\xi_1 = \xi_2 = 0} = (-v_3^1 F_{\xi_1 \xi_1})_{\xi_1 = \xi_2 = 0}.$$

Thus, (14) for $\underline{v}$ becomes:

$$\int_\Omega (\operatorname{rot} \mathbf{v})^2 \, dx = \int_\Omega \sum_{i,\, k=1}^{3} \left(\frac{\partial v_i}{\partial x_k} \right)^2 dx - \int_S \mathbf{v}^2 (F_{\xi_1 \xi_1} + F_{\xi_2 \xi_2}) \, dS. \qquad (20)$$

Let us evaluate the surface integral. Let $a_k(x)$ be functions continuously-differentiable in $\overline{\Omega}$, which satisfy the condition:

$$a_k |_S = n_k \quad (k = 1, 2, 3).$$

Then[1]:

[1]This method of evaluating limit integrals in terms of integrals over an area has been borrowed from O. A. Ladyzhenskaya, who used it to evaluate the derivatives of a function in terms of the degrees of its elliptical operator.

$$\int_S \mathbf{v}^2 (F_{\xi_1\xi_1} + F_{\xi_2\xi_2})\, dS \leqslant M \int_S \mathbf{v}^2 \sum_{k=1}^{3}{}' a_k n_k\, dS = M \sum_{k=1}^{3} \int_\Omega \frac{\partial}{\partial x_k} (a_k \mathbf{v}^2)\, dx =$$

$$= M \sum_{k=1}^{3} \int_\Omega \left(\frac{\partial a_k}{\partial x_k} \mathbf{v}^2 + a_k \frac{\partial \mathbf{v}^2}{\partial x_k} \right) dx \leqslant M_1 M \left[\delta \int_\Omega \sum_{i,\,k=1}^{3} \left(\frac{\partial v_i}{\partial x_k} \right)^2 dx + \right. \tag{21}$$

$$\left. + \left(1 + \frac{1}{\delta}\right) \int_\Omega \mathbf{v}^2 dx \right],$$

where:

$$M = \max |F_{\xi_1\xi_1} + F_{\xi_2\xi_2}|, \; M_1 = \max_{x \in \Omega} \left\{ |a_k(x)|, \left| \frac{\partial a_k}{\partial x_i} \right| \right\},$$

δ is an arbitrary positive number.

In view of this evaluation, we obtain from (21) the inequality:

$$\int_\Omega \sum_{i,\,k=1}^{3} \left(\frac{\partial v_i}{\partial x_k} \right)^2 dx \leqslant \int_\Omega (\operatorname{rot} \mathbf{v})^2 dx +$$

$$+ MM_1 \left[\delta \int_\Omega \sum_{i,\,k=1}^{3} \left(\frac{\partial v_i}{\partial x_k} \right)^2 dx + \left(1 + \frac{1}{\delta}\right) \int_\Omega \mathbf{v}^2 dx \right].$$

Taking δ sufficiently small and evaluating the second term in brackets by inequality (17), we in fact obtain the requisite inequality (16).

Remark. The full proof of this lemma could have been based directly on the expression for $\underline{v}$ in terms of rot$\underline{v}$, given in lemma 1.2.

Lemma 5.2. For smooth vectors $\underline{u}$ in Ω, which have div$\underline{u}$ = 0, $u_n|_S = 0$, there obtains the inequality:

$$\|\mathbf{u}\|_{W_2^1} \leqslant C \|\operatorname{rot} \mathbf{u}\|_{L_2}. \tag{22}$$

Let us first prove the inequality:

$$\int_\Omega \mathbf{u}^2 dx \leqslant C \int_\Omega (\mathrm{rot}\,\mathbf{u})^2\, dx. \tag{23}$$

Making use of lemma 1.2, we can represent $\underline{u}$ in the form:

$$\mathbf{u} = \mathrm{rot}\,\mathbf{v},\ (\mathrm{div}\,\mathbf{v} = 0,\ v_\tau|_S = 0).$$

Then rot$\underline{u}$ = $-\Delta\underline{v}$ and inequality (23) is equivalent to the following:

$$\int_\Omega (\mathrm{rot}\,\mathbf{v})^2\, dx \leqslant C \int_\Omega (\Delta\mathbf{v})^2\, dx. \tag{24}$$

Let us prove it. We have:

$$\int_\Omega (\mathrm{rot}\,\mathbf{v})^2\, dx = -\int_\Omega \mathbf{v}\cdot\Delta\mathbf{v}\, dx, \tag{25}$$

from which:

$$\int_\Omega (\mathrm{rot}\,\mathbf{v})^2\, dx \leqslant \left(\int_\Omega \mathbf{v}^2 dx\right)^{1/2} \left(\int_\Omega (\Delta\mathbf{v})^2\, dx\right)^{1/2}.$$

Paying attention to (17) we have:

$$\int_\Omega (\mathrm{rot}\,\mathbf{v})^2\, dx \leqslant C\left(\int_\Omega (\mathrm{rot}\,\mathbf{v})^2\, dx\right)^{1/2} \left(\int_\Omega (\Delta\mathbf{v})^2\, dx\right)^{1/2},$$

whence we in fact have (24) and, along with it, (23) as well.

Let us move on to evaluate the derivatives of $\underline{u}$.

Similar to lemma 4.2, we shall show that in (14), as applied to $\underline{u}$, the integral over S does not contain derivatives of $\underline{u}$. Let u_1^1, u_2^1, u_3^1 be components of $\underline{u}$ in coordinates (ξ_1, ξ_2, ξ_3), introduced in lemma 4.2; for convenience the tangential coordinate ξ_1 is chosen to coincide by direction with $\underline{u}_\tau$, so

that:

$$u_2^1|_0=0.$$

Let ℓ be the section of S intersecting the plane $\xi_1 O \xi_3$, while the point O lies on ℓ. /2(

We have:

$$\mathbf{u}\times \operatorname{rot}\mathbf{u}\cdot\mathbf{n}=u_1^1\left(\frac{\partial u_3^1}{\partial \xi_1}-\frac{\partial u_1^1}{\partial \xi_3}\right)-u_2^1\left(\frac{\partial u_2^1}{\partial \xi_3}-\frac{\partial u_3^1}{\partial \xi_2}\right)=u_1^1\left(\frac{\partial u_3^1}{\partial \xi_1}-\frac{\partial u_1^1}{\partial \xi_3}\right),$$

$$\frac{1}{2}\operatorname{grad}\mathbf{u}^2\cdot\mathbf{n}=u_1^1\frac{\partial u_1^1}{\partial \xi_3}.$$

Consequently: $$\mathbf{u}\times\operatorname{rot}\mathbf{u}\cdot\mathbf{n}+\frac{1}{2}\operatorname{grad}\mathbf{u}^2\cdot\mathbf{n}=u_1^1\frac{\partial u_3^1}{\partial \xi_1}.$$

But:

$$u_3^1=\bar{u}_1\cos(\bar{\xi}_1\xi_3)+\bar{u}_2\cos(\bar{\xi}_2,\ \xi_3)+\bar{u}_3\cos(\bar{\xi}_3,\ \xi_3),$$

from which:

$$\left.\frac{\partial u_3^1}{\partial \xi_1}\right|_0=\ u_1^1\frac{\partial}{\partial \xi_1}\cos(\bar{\xi}_1,\ \xi_3).$$

Thus, identity (14) for u becomes:

$$\int_\Omega(\operatorname{rot}\mathbf{u})^2\,dx=\int_\Omega\sum_{i,\,k=1}^{3}\left(\frac{\partial u_i}{dx_k}\right)^2dx-\int_S\mathbf{u}^2\frac{\partial}{\partial \bar{\xi}_1}\cos(\bar{\xi}_1,\ \xi_3)\,dS. \qquad (26)$$

The further considerations are entirely analogous to lemma (4.2).

Remark. It is curious that the surface integrals in identities (20) and (26) contain terms with sign determined by the direction of convexity of the surface S. If S is con-

vex, they are nonpositive, and then from (20) and (26) it follows that:

$$\int_\Omega \sum_{i,k=1}^{3} \left(\frac{\partial v_i}{dx_k}\right)^2 dx \leqslant \int_\Omega (\operatorname{rot} \mathbf{v})^2 \, dx$$

as well as an identical inequality for $\underline{u}$.

This enables perfectly elementary evaluations (17) and (23) for convex regions, as

$$\int_\Omega \mathbf{v}^2 dx$$

can be evaluated in terms of the Dirichlet integral, similar to the case of scalar functions [12].

III. Theorems Concerning Representation of Vectors of J and $\overset{\circ}{J}$ in the Form of Curls

Theorem 3.2 follows immediately from the definition of $\overset{\circ}{J}$ and lemmas 1.2 and 4.2.

Theorem 3.2. Any vector $\mathbf{u} \in \overset{\circ}{J}$ can be represented as $\underline{u} = \operatorname{rot}\underline{v}$, where:

$$\mathbf{v} \in W_2^1(\Omega); \; \operatorname{div} \mathbf{v} = 0; \; v_\tau|_S = 0,$$

while:

$$\|\mathbf{v}\|_{W_2^1} \leqslant C \|\mathbf{u}\|_{L_2}.$$

In a similar fashion, theorem 4.2 follows from the definition of J and lemmas 2.2 and 5.2.

Theorem 4.2. Any vector $\mathbf{w} \in J$ can be represented as $w = \operatorname{rot}\underline{u}$, where:

$$\mathbf{u} \in W_2^1(\Omega); \; \operatorname{div} \mathbf{u} = 0; \; u_n|_S = 0,$$

while:

$$\|\mathbf{u}\|_{W_2^1} \leqslant C\|\mathbf{w}\|_{L_2}.$$

Remark. By using the expressions for solenoidal vectors $\underline{v}$ and $\underline{u}$, which have:

$$v_\tau|_S=0, \quad u_n|_S=0$$

in terms of their rot (lemmas 1.2 and 2.2), it could be proved (cf. [3]) that, if rot$\underline{v}$ or rot$\underline{u}$ is in $W_2^n(\Omega)$, then $\underline{v}$ and $\underline{u}$ are in $W_2^{n+1}(\Omega)$ and the inequality obtains:

$$\|\mathbf{v}\|_{W_2^{n+1}} \leqslant C\|\mathrm{rot}\,\mathbf{v}\|_{W_2^n}$$

with a similar one for $\underline{u}$ (the boundary of S is presumed to be n + 2 times continuously-differentiable). Since we shall not use this fact either in the present or subsequent works, we do not present its proof. /21

§3. Equivalents of the Subspaces $\overset{\circ}{G}$, J, $\overset{\circ}{J}$, U, G to the Subspaces of Weyl in Chapter I

Theorem 5.2. Subspaces $\overset{\circ}{G}$, J, $\overset{\circ}{J}$, U, G, introduced in §1 of this chapter, concur with the identically-designated subspaces of Chapter I.

In this theorem we shall furnish the symbol ^ in denoting all the subspaces introduced at the outset of the present chapter, to distinguish them from the subspaces of Chapter I. The concurment of $\overset{\circ}{G}$ and $\overset{\circ}{G}$[1] follows directly from the fact that, as is known, the closure of vectors of type gradϕ, where:

$$\varphi|_S=0,$$

---

[1] Evidently the symbol ^ has been omitted [Tr].

coincides with the set of gradients of functions in the closure of ϕ with respect to the norm of $W_2^1(\Omega)$, while the latter, as is well known [21], can be obtained from ϕ which vanish in the boundary zones.

We shall show that $\hat{J} = \mathring{J}$.

It follows directly from the definition of $\mathring{J}$ and $\hat{J}$ that:

$$\hat{J} \subset \mathring{J},$$

for the curls of finite vectors are contained in the lineal J, the closure of which produces $\mathring{J}$. Therefore it is sufficient to show that any vector $\underline{u}$ of J can be approximated with respect to the norm of $L_2(\Omega)$ by vectors of the type $\mathrm{rot}\underline{v}^{(n)}$, where $\underline{v}^{(n)}$ are finite, with any desired accuracy.

Using lemma 1.2, we shall represent $\underline{u}$ as $\mathrm{rot}\underline{v}$, where $\underline{v}$ is a smooth vector having $v_\tau|_S = 0$.

Let $\zeta_n(x)$ be a cut function that is continuous and continuously differentiable, equaling 0 in a zone Ω_ε of width ε_n about S, equaling 1 outside the zone $\Omega_{2\varepsilon}$ of width $2\varepsilon_n$, and satisfying the inequalities:

$$|\zeta_n(x)| \leqslant 1; \quad |\mathrm{grad}\,\zeta_n(x)| \leqslant \frac{C}{\varepsilon_n}.$$

Moreover, we may regard $\zeta_n(x)$ as constant on surfaces parallel to S.

We shall show that $\zeta(x)\underline{v}$ can be used as $\underline{v}^{(n)}$. We have:

$$\int_{\Omega} (\mathrm{rot}\,\mathbf{v} - \mathrm{rot}\,\zeta_n \mathbf{v})^2\, dx = \int_{\Omega_\varepsilon} (\mathrm{rot}\,\mathbf{v})^2\, dx + \int_{\Omega_{2\varepsilon}-\Omega_\varepsilon} (\mathrm{rot}\,\mathbf{v} - \mathrm{rot}\,\zeta_n \mathbf{v})^2\, dx =$$

$$= \int_{\Omega_\varepsilon} (\mathrm{rot}\,\mathbf{v})^2\, dx + \int_{\Omega_{2\varepsilon}-\Omega_\varepsilon} (\mathrm{rot}\,\mathbf{v} - \zeta_n\,\mathrm{rot}\,\mathbf{v} - \mathrm{grad}\,\zeta_n \times \mathbf{v})^2\, dx \leqslant$$

$$\leqslant \int_{\Omega_\varepsilon} (\mathrm{rot}\,\mathbf{v})^2\, dx + C \int'_{\Omega_{2\varepsilon}-\Omega_\varepsilon} (\mathrm{rot}\,\mathbf{v})^2\, dx + 2 \int_{\Omega_{2\varepsilon}-\Omega_\varepsilon} |\mathrm{grad}\,\zeta_n \times \mathbf{v}|^2\, dx.$$

Let $n \to \infty$ and $\varepsilon_n \to 0$. The first two terms on the right tend to zero. Let us examine the third. Using the condition of constancy of $\zeta_n(x)$ on surfaces parallel to S, and considering the condition:

$$v_\tau|_S=0,$$

we have:

$$|\operatorname{grad}\zeta_n \times \mathbf{v}|^2=|\operatorname{grad}\zeta_n|^2|v_\tau|^2 \leqslant K|\operatorname{grad}\zeta_n|^2\varepsilon_n^2 \leqslant KC,$$

where the constant K is determined by the properties of S and $\max\limits_{x\in\Omega}\left\{|\mathbf{v}|,\ \left|\frac{\partial v_i}{\partial x_k}\right|\right\}$

Therefore the third term also tends to zero, so that: /22

$$\operatorname{rot}\zeta_n\mathbf{v}\xrightarrow{L_2(\Omega)}\operatorname{rot}\mathbf{v}$$

and the coincidence of $\overset{\circ}{J}$ and $\hat{J}$ is proved.

The concurment of all the other subspaces now becomes obvious by comparing resolutions (7) and (11): J and $\hat{J}$ coincide as complements to $\overset{\circ}{G} = \hat{\overset{\circ}{G}}$; G and $\hat{G}$ as complements to $\overset{\circ}{J} = J$,[1] while U and U as complements to:

$$G\ominus J=\overset{\circ}{G}\ominus J.$$

§4. The Properties of Projectors into the Weyl Subspaces

Theorem 6.2. If the vector:

$$\mathbf{u}\in W_2^n(\Omega),$$

---

[1]Unfortunately parts of the symbols may have been obliterated in the original copy, and this paragraph should be checked against the original [Tr].

then its projections into subspaces $\overset{\circ}{G}$, J, U, $\overset{\circ}{J}$, G also belong to $W_2^n(\Omega)$, and for each such projection P$\underline{u}$ the inequality is valid:

$$\|P\mathbf{u}\|_{W_2^n} \leqslant C\|\mathbf{u}\|_{W_2^n}.$$

The boundary of the region in this case is presumed to be sufficiently smooth.[1]

The theorem will be proven if the requisite inequality is established for smooth $\underline{u}$, for then any vector:

$$\mathbf{u} \in W_2^n(\Omega)$$

can be approximated by smooth vectors with respect to the norm of this space, followed by a passage to the limit in the inequality. It is sufficient to restrict our treatment to projectors onto $\overset{\circ}{G}$ and $\overset{\circ}{J}$, for it follows from resolutions (11) that:

$$P_J = I - P_{\overset{\circ}{G}};\ P_G = I - P_{\overset{\circ}{J}};\ P_U = I - P_{\overset{\circ}{G}} - P_J,$$

where I is the identity transformation.

For the proof let us recall the connection between operations $P_{\overset{\circ}{G}}$ and $P_{\overset{\circ}{J}}$ and the solution of boundary-value problems (theorem 1.2). If a smooth vector $\underline{u}$ is given, then:

$$P_{\overset{\circ}{G}}\mathbf{u} = \operatorname{grad}\varphi,$$

where φ is the solution of the problem:

$$\Delta\varphi = \operatorname{div}\mathbf{u},$$
$$\varphi|_S = 0,$$

---

[1]An analysis of the findings of the cited works shows that it is sufficient for the boundary to be n + 1 times continuously differentiable.

while:

$$P_j\mathbf{u}=\mathbf{u}_3^1+\operatorname{grad}\psi,\quad \text{where}\quad \mathbf{u}_3^1=\operatorname{rot}\frac{1}{4\pi}\int_{E_3}\frac{\operatorname{rot}\mathbf{u}}{|x-y|}dy$$

($\underline{u}$ is continued beyond Ω so that the continuation is finite); ψ is the solution of the problem:

$$\Delta\psi=0,\ \frac{\partial\psi}{\partial n}\Big|_S=-u_{3n}^1\Big|_S.$$

Thus, the proof of the theorem reduces to evaluations involving boundary-value problems and the Newtonian potential. It is in fact sufficient to prove that:

$$\|\mathbf{u}_3^1\|_{W_2^n}\leqslant C\|\mathbf{u}\|_{W_2^n}, \tag{27}$$

$$\|\operatorname{grad}\varphi\|_{W_2^n}\leqslant C\|\operatorname{div}\mathbf{u}\|_{W_2^{n-1}}, \tag{28}$$

$$\|\operatorname{grad}\psi\|_{W_2^n}\leqslant C\|\mathbf{u}\|_{W_2^n}. \tag{29}$$

a) The finite prolongation of $\underline{u}$ beyond Ω can be arranged so as to satisfy the inequality:

$$\|\mathbf{u}\|_{W_2^n(E_3)}\leqslant C\|\mathbf{u}\|_{W_2^n(\Omega)} \tag{30a}$$

with constant C independent of the function being prolonged. /23

Such a property is possessed, e.g., by a prolongation using the construction of Hesten's, employed in the work of V. M. Babich [1].[1]

If we compare the form of $\underline{u}_3^1$ with formula (1) of Chapter I,

[1]Fulfillment of this inequality in [1] is not specially mentioned, but an analysis of the construction enables its establishment (such an analysis had to be made in [3] by one of the authors).

it is evident that $\underline{u}_3^1$ is the projection of the finite prolongation of $\underline{u}$ onto $J(E_3)$. Since the finite prolongation of $\underline{u}$ belongs to $W_2^n(E_3)$, then by theorem 1.1 and

$$\mathbf{u}_3^1 \in W_2^n(E_3),$$

while:

$$\|\mathbf{u}_3^1\|_{W_2^n(E_3)} \leqslant \|\mathbf{u}\|_{W_2^n(E_3)},$$

we have:

$$\|\mathbf{u}_3^1\|_{W_2^n(\Omega)} \leqslant \|\mathbf{u}_3^1\|_{W_2^n(E_3)} \leqslant \|\mathbf{u}\|_{W_2^n(E_3)} \leqslant C\|\mathbf{u}\|_{W_2^n(\Omega)}.$$

Here inequality (30a) was used at the end.

b) An evaluation of type (28) had also been used for a more general elliptical equation in a number of works.

If the function ϕ satisfied, besides the condition:

$$\varphi|_S = 0$$

also conditions:

$$\Delta\varphi|_S = 0,\ \Delta^2\varphi|_S = 0$$

and so on up to a certain order determined by the number "n"[1], it would be sufficient to refer to Chapter II of the book by O. A. Ladyzhenskaya [14]. These missing boundary conditions can be obtained in the same way as described in her article [15].

c) Estimate (29) is proved in similar fashion to (28) if we consider estimate (27). For $u_{3n}^1|_S$ we construct the function:

$$\tilde{\varphi} \in W_2^{n+1}(\Omega),$$

[1] Up to order $\Delta^{\left[\frac{n}{2}\right]}\varphi|_S = 0.$

satisfying the inequality:

$$\|\tilde{\Phi}\|_{W_2^{n+1}} \leqslant C\|\mathbf{u}_3^1\|_{W_2^n},$$

and such that $\psi-\tilde{\psi}$ satisfies conditions:

$$\left.\frac{\partial(\psi-\tilde{\psi})}{\partial n}\right|_S=0;\ \left.\frac{\partial\Delta(\psi-\tilde{\psi})}{\partial n}\right|_S=0$$

and so forth. The possibility of this construction follows from [19] and [20]. Then, for an estimate of $\psi-\tilde{\psi}$ we may again refer to [14], and for ψ we obtain the estimate:

$$\|\psi\|_{W_2^{n+1}} \leqslant C\|\mathbf{u}_3^1\|_{W_2^n}$$

and moreover:

$$\|\operatorname{grad}\psi\|_{W_2^n(\Omega)} \leqslant C\|\mathbf{u}_3^1\|_{W_2^n(\Omega)}.$$

Taking heed of (27) we in fact obtain (29).

Chapter III. Multiply-Connected and Unbounded Regions

§1. A Bounded Multiply-Connected Region

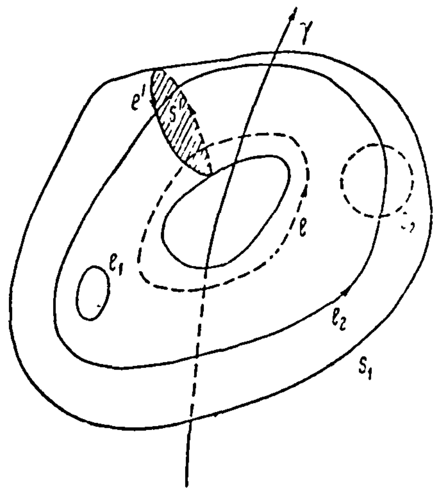
There are two characteristics of a multiply-connected region Ω that are important hereafter (cf. figure).

1. The existence in it of closed contours ℓ which cannot be deformed into a point still inside Ω.

2. The existence of closed surfaces Σ with a similar property.

To be specific we shall consider an elementary region that is multiply-connected in both respects, namely a region bounded by two surfaces: S_1, a homeomorphic toroidal surface, and S_2,

a homeomorphic spherical surface placed inside the torus. We shall denote the entire surface $S_1 + S_2$ by $\tilde{S}$. This region evidently possesses one contour of type ℓ and one surface of type Σ. We shall introduce some definitions. /24



Let ℓ' be one of two closed contours on S_1 that cannot be reduced to a point by continuous deformation on S_1, over which there may extend a surface lying entirely within Ω. We shall denote this surface by S'.

If the circulation of a certain vector $\underline{u}$ along a contour of type ℓ does not depend on the choice of ℓ, which obtains when $\underline{u}$ = gradf, while f is a multi-valued function that varies by a constant in making the circuit of ℓ, this circulation is known as the univariate period of $\underline{u}$ and denoted by:

$$C^{\prime}[\mathbf{u}]=\oint \mathbf{u}\cdot d\mathbf{l}.$$

If $\underline{u}$ = grad f, we shall also write:

$$C^{\prime}[\mathbf{u}]=C^{\prime}[f].$$

Let Σ be a closed surface within Ω, containing S_2 inside itself.

If the flux of the vector $\underline{u}$ does not depend on the choice of Σ, which obtains when $\operatorname{div}\underline{u} = 0$, this flux is known as the bivariate period of $\underline{u}$ and denoted by:

$$C^2[\mathbf{u}] = \int_{\Sigma} u_n d\Sigma.$$

If, further, $\underline{u} = \operatorname{grad} f$, we shall write:

$$C^2[f] = C^2[\operatorname{grad} f]$$

such that:

$$C^2[f] = \int_{\Sigma} \frac{\partial f}{\partial n} d\Sigma.$$

Let us identify in Ω two standard harmonic functions, each of which generates a unidimensional subspace in the orthogonal resolution of L_2.

Lemma 1.3. Let h_1 be a function harmonic in Ω, satisfying boundary conditions:

$$h_1|_{S_1} = 0; \quad h_1|_{S_2} = \text{const} \neq 0.$$

Then: $C^2[h_1] \neq 0.$

Regarding S as sufficiently smooth, we may also affirm the smoothness of h_1 up to S inclusive. We have:

$$\int_{\Omega} (\operatorname{grad} h_1)^2 dx = -\int_{\Omega} h_1 \Delta h_1 dx + \int_{S_1+S_2} h_1 \frac{\partial h_1}{\partial n} dS = \text{const} \int_{S_2} \frac{\partial h_1}{\partial u} dS.$$

Since h_1 departs from a constant in Ω:

$$\int_{S_2} \frac{\partial h_1}{\partial n} dS = \int_{S_2} (\operatorname{grad} h_1)_n dS \neq 0$$

and by virtue of the solenoidal character of $\operatorname{grad} h_1$:

$$\int_{\Sigma} (\operatorname{grad} h_1)_n d\Sigma = \int_{S_2} (\operatorname{grad} h_1)_n dS \neq 0.$$

Lemma 2.3. In Ω there exists a multiple-valued harmonic function h_2 with the following properties:

$$1)\ C^1[h_2] \neq 0;$$

$$2)\ \left.\frac{\partial h_2}{\partial n}\right|_S = 0;$$

$$3)\ \int\limits_{S'} \frac{\partial h_2}{\partial n} dS' \neq 0.$$

In order to satisfy requirement 1 it is enough to take the scalar potential h_2' of the magnetic field intensity of a steady current traveling along a conductor γ that is passed through a "hole" in the region bounded by the surface S_1 (cf., e.g., [10]). It is determined from the Biot-Savart formula: /25

$$\operatorname{grad} h_2'(x) = \int\limits_{\gamma_y} \frac{(\mathbf{x}-\mathbf{y}) \times d\gamma_y}{|x-y|^3}.$$

We note that, although h_2' is a multi-valued function, $\operatorname{grad} h_2'$ is obviously a single-valued vector, for h_2' changes by a constant in making the circuit of ℓ.

In order to satisfy condition 2, as well as condition 1, we must add to h_2' the solution (single-valued) of the problem:

$$\Delta h_2 = 0$$

in Ω:

$$\left.\frac{\partial h_2''}{\partial n}\right|_S = -(\operatorname{grad} h_2')_n \Big|_{S}.\text{*}$$

Then it is easy to show that condition 3 is a consequence of conditions 1 and 2.

In fact, let S' produce a sectioning of region Ω, con-

---

[1]Apparently the condition of solvability of this problem is fulfilled.

verting it into region Ω', lacking contours of type ℓ, and the function h_2 into a single-valued function.

As a result of integration by parts in region Ω' we have:

$$\int_{\Omega} \operatorname{grad}^2 h_2 dx = -\int_{\Omega'} h_2 \Delta h_2 dx + \int_{S_1+S_2} h_2 \frac{\partial h_2}{\partial n} dS + \int_{S'} (h_2^+ - h_2^-) \frac{\partial h_2}{\partial n} dS',$$

where h_2^+ and h_2^- are values of h_2 at different sides of S'. Since:

$$\Delta h_2 = 0; \ \left.\frac{\partial h_2}{\partial n}\right|_S = 0 \text{ and } h_2^+ - h_2^- = C^1[h_2],$$

we have:

$$\int_{\Omega} (\operatorname{grad} h_2)^2 dx = C^1[h_2] \int_{S'} \frac{\partial h_2}{\partial n} dS'.$$

Here the left side does not vanish, for h_2 is not constant, in view of condition 1. Consequently:

$$\int_{S'} \frac{\partial h_2}{\partial n} dS' \neq 0.$$

Q.E.D.

Remark. Multiplying $h_2(x)$ by a constant other than zero does not alter its properties as given in the lemma. We shall consider that $h_2(x)$ is normalized so that:

$$C^1[h_2] = 1.$$

Let us identify in L_2 the lineals of mutually-orthogonal smooth vectors: $\tilde{G}$ vectors of type gradϕ:

$$\varphi|_s = 0,$$

$\tilde{J}$, smooth solenoidal vectors (the orthogonality of which is easily checked). We can identify in $\tilde{J}$ orthogonal lineals $\tilde{U}_1$ or vectors of type αgradh_1, where α is a constant, and h_1 is

the function appearing in lemma 1.3. $\tilde{J}'$ are solenoidal vectors $\underline{w}$ for which: /26

$$C^2[\mathbf{w}]=0.$$

We shall demonstrate the orthogonality of these lineals. Since:

$$C^2[\mathbf{w}]=\int_{\Sigma} w_n d\Sigma,$$

we also have:

$$\int_{S_2} w_n dS=0$$

in view of the solenoidal nature of $\underline{w}$ between Σ and S_2.

Consequently:

$$\int_{\Omega} \operatorname{grad} h_1 \cdot \mathbf{w} dx=\int_{\Omega} h_1 \operatorname{div} \mathbf{w} dx+\int_{S_1} h_1 w_n dS+$$
$$+\int_{S_2} h_1 w_n dS=h_1\Big|_{S_2}\int_{S_2} w_n dS=0.$$

In $\tilde{J}'$ we can identify orthogonal lineals: $\tilde{U}'$ or vectors of the type gradh, where h are single-valued harmonic functions that are smooth in Ω and have:

$$C^2[h]=0.$$

$\tilde{j}'$ are smooth solenoidal vectors $\underline{u}$ for which:

$$u_n|_S=0.$$

Their orthogonality is easily verified.

In $\tilde{j}'$ we can identify the orthogonal lineals: $\tilde{U}_2$ or vectors of type βgradh_2, where β is a constant, and h_2 is the function in lemma 2.3. $\tilde{j}$. are smooth solenoidal vectors $\underline{u}$ for which:

$$u_n|_S=0 \quad \text{and} \quad \int\limits_{S'} u_n dS'=0.$$

The orthogonality of these lineals emerges from the following. Let, as in lemma 2.3, S' produce a sectioning of the region Ω. Using the definitions of lemma 2.3 we have:

$$\int\limits_{\Omega} \operatorname{grad} h_2 \cdot \mathbf{u} dx = \int\limits_{\Omega'} \operatorname{grad} h_2 \cdot \mathbf{u} dx = \int\limits_{\Omega'} h_2 \operatorname{div} \mathbf{u} dx +$$

$$+ \int\limits_{S} h_2 u_n dS + \int\limits_{S'} (h_2^+ - h_2^-) u_n dS.$$

The first two terms on the right vanish thanks to the properties of u, while the third is:

$$C^1[h_2] \int\limits_{S'} u_n dS'$$

and, consequently, is also equal to zero.

We shall denote the closures in L_2 of the above-introduced lineals by the identical symbols without the tilde.

Theorem 1.3. The following orthogonal resolutions obtain:

$$\mathring{J}' = U_2 \oplus \mathring{J}, \tag{30b}$$
$$J' = U' \oplus \mathring{J}', \tag{31}$$
$$J = U_1 \oplus J', \tag{32}$$
$$L_2 = \mathring{G} \oplus J, \tag{33}$$

which also implies:

$$L_2 = \mathring{G} \oplus U_1 \oplus U' \oplus U_2 \oplus \mathring{J}. \tag{34}$$

It is sufficient to prove that the analogous resolutions /27
obtain for the vectors of $\mathring{J}', 'J'$ and so forth, and then to close them in L_2.

1. Let us prove resolution (30b), for which it is sufficient to show that $\mathbf{u} \in \mathring{J}'$ can be represented as:

$$\mathbf{u} = \mathbf{u}_{\tilde{G}_2} + \mathbf{u}_J.$$

It is sufficient to assume:

$$\mathbf{u}_{\tilde{G}_2} = \frac{\int\limits_{S'} u_n dS'}{\int\limits_{S'} (\operatorname{grad} h_2)_n \, dS'} \operatorname{grad} h_2.$$

Then:

$$\mathbf{u} - \mathbf{u}_{\tilde{G}_2} \in \mathring{J},$$

for:

$$\int\limits_{S'} (\mathbf{u} - \mathbf{u}_{\tilde{G}_2})_n dS' = \int\limits_{S'} u_n dS' - \frac{\int\limits_{S'} u_n dS'}{\int\limits_{S'} (\operatorname{grad} h_2)_n \, dS'} \int\limits_{S'} (\operatorname{grad} h_2)_n dS' = 0.$$

2. Let $\mathbf{u} \in \mathring{J}'$ We shall prove that:

$$\mathbf{u} = \mathbf{u}_{\tilde{G}'} + \mathbf{u}_J.$$

At first, exactly as in theorem 1.2 of Chapter II, we solve the problem:

$$\operatorname{rot} \mathbf{v} = \operatorname{rot} \mathbf{u},$$
$$\operatorname{div} \mathbf{v} = 0,$$
$$v_n |_S = 0.$$

For this it is sufficient, as in that place, to construct at first the vector $\underline{v}'$ in the following manner, satisfying the conditions of this problem except for the last: $\underline{u}$ is continued beyond Ω while preserving smoothness and vanishing outside the region Ω', somewhat wider than Ω, and then assume that:

$$\mathbf{v}' = \frac{1}{4\pi} \int\limits_{\Omega'} \frac{\operatorname{rot} \mathbf{u}(y)}{|x - y|} dy.$$

Then:

$$\mathbf{v}=\mathbf{v}'+\operatorname{grad}\psi,$$

where ψ is the solution of the Neuman problem:

$$\Delta\psi=0 \text{ в } \Omega;\ \frac{\partial\psi}{\partial n}\Big|_S=-v_n'|_S.$$

Then, since:

$$\operatorname{rot}(\mathbf{u}-\mathbf{v})=0 \text{ and } \operatorname{div}(\mathbf{u}-\mathbf{v})=0,$$

we obtain:

$$\mathbf{u}-\mathbf{v}=\operatorname{grad}h',$$

where h' is a harmonic function for which, in general, $C^1[\bar{h}']\neq 0$.

Let us assume that:

$$h'=h+C^1[h']h_2.$$

Then h is a single-valued harmonic function.

Thus:

$$\mathbf{u}=\operatorname{grad}h+C^1[h']\operatorname{grad}h_2+\mathbf{v}. \tag{35}$$

The vector:

$$\mathbf{v}''\equiv C^1[h']\operatorname{grad}h_2+\mathbf{v}\in\tilde{J}',$$

which means that:

$$C^2[\mathbf{v}'']=0,$$

and since $C^2[\underline{u}] = 0$ as well (for $\mathbf{u}\in\tilde{J}'$), it is also true that:

$$C^2[\operatorname{grad}h]=0, \text{ i.e. } \operatorname{grad}h\in\bar{U}.$$

Thus, (35) is in fact the resolution (31) to be proved in this

section, in which:

$$\mathbf{u}_{\mathcal{L}'} = \operatorname{grad} h, \quad \mathbf{u}_{J'} = \mathbf{v}''.$$

3. Let $\mathbf{u} \in J$. We shall prove that: /28

$$\mathbf{u} = \mathbf{u}_{\mathcal{L}_1} + \mathbf{u}_{J'}. \tag{36}$$

Let us consider the vector:

$$\mathbf{u} - C^2[\mathbf{u}] \operatorname{grad} h_1.$$

It is solenoidal and its bidimensional cycle is equal to zero, i.e. it belongs to $\tilde{J}'$. Consequently (36) holds, while:

$$\mathbf{u}_{\mathcal{L}_1} = C^2[\mathbf{u}] \operatorname{grad} h_1.$$

4. Let $\underline{u}$ be any smooth vector. We shall prove that:

$$\mathbf{u} = \mathbf{u}_{\tilde{G}} + \mathbf{u}_{J}.$$

Since $\underline{u}_{\tilde{G}}$ should have the form gradϕ:

$$\varphi|_S = 0$$

and we should have:

$$\operatorname{div} \mathbf{u}_J = \operatorname{div}(\mathbf{u} - \mathbf{u}_{\tilde{G}}) = 0,$$

then to determine ϕ we arrive at the problem:

$$\Delta\varphi = 0, \quad \varphi|_S = 0,$$

by solving which we obtain:

$$\mathbf{u}_{\tilde{G}} = \operatorname{grad} \varphi.$$

$\underline{u}_{\tilde{J}}$ is determined as $\underline{u}$-gradϕ. Q.E.D.

Remark. In view of the unidimensional nature of lineals $\tilde{U}_1$ and $\tilde{U}_2$ it is obvious that:

$$U_1 = \tilde{U}_1 \quad \text{and} \quad U_2 = \tilde{U}_2.$$

For smooth vectors belonging to the introduced lineals there hold analogues of lemmas 1.2, 2.2 and 3.2 of Chapter II.

Lemma 3.2 is literally valid as well for a multiply-connected region, as the topological features of the boundary did not figure in its proof.

Lemma 3.3. In order for a smooth vector u to be capable of representation as u = rotv, where v satisfies condition:

$$\mathbf{v}_{\tau}|_{S}=0,$$

it is necessary and sufficient that it belong to the lineal $\tilde{J}_1$, and if this is so we can select v from $\tilde{J}'$.

Necessary condition. If:

$$\mathbf{v}_{\tau}|_{S}=0,$$

then v is a solenoidal vector and:

$$(\operatorname{rot}\mathbf{v})_n|_{S}=0.$$

It remains to show that:

$$\int_{S'}(\operatorname{rot}\mathbf{v})_n dS'=0.$$

By the formula of Stokes:

$$\int_{S'}(\operatorname{rot}\mathbf{v})_n dS'=\int_{l'}\mathbf{v}\cdot d\mathbf{l}=\int_{l'}\mathbf{v}_{\tau}\cdot d\mathbf{l}=0.$$

(ℓ' is a contour on S_1 that embraces S').

Sufficient condition. Let $\mathbf{u}\in\tilde{J}$. In exactly the same way as lemma 1.2, we construct vector $\mathbf{v}_1$, for which:

$$\operatorname{div}\mathbf{v}_1=0 \text{ and } \operatorname{rot}\mathbf{v}_1=\mathbf{u}.$$

We now construct $\mathbf{v}_{II}$, for which:

$$\operatorname{div} \mathbf{v}_{II} = 0, \ \operatorname{rot} \mathbf{v}_{II} = 0; \ v_{II\tau}|_S = -v_{I\tau}|_S.$$

For this, it is sufficient to find a harmonic function ϕ, even with:

$$C^1[\varphi] \neq 0,$$

such that:

$$(\operatorname{grad} \varphi)_\tau|_S = -v_{I\tau}|_S,$$

and to set $\underline{v}_{II} = \operatorname{grad}\phi$. If ℓ_1 is such a closed contour on S_1 or S_2 that can be drawn up into a point while remaining on S, then the same as in lemma 1.2:

$$\int_{l_1} \mathbf{v}_{I\tau} \mathbf{dl} = 0.$$

Thus the values of $\mathbf{v}_{I\tau}$ on S_2 may be regarded as the gradient of a certain function ϕ_{02} given on S_2. On S_1 we can identify two types of closed contours that do not roll up into a point. One of these is ℓ', the other ℓ_2 (figure). /29

$$\oint_{l'} \mathbf{v}_{I\tau} \cdot \mathbf{dl} = \oint_{l'} \mathbf{v}_I \mathbf{dl} = \int_{S'} (\operatorname{rot} \mathbf{v}_I)_n dS' = \int_{S'} u_n dS' = 0.$$

In regard to:

$$\int_{l_2} \mathbf{v}_{I\tau} \mathbf{dl},$$

in general it is different from zero.

Thus, $\underline{v}_{I\tau}$ is the gradient of the function ϕ_{01} given on S_1, ϕ_{01} changing by a constant in making the circuit of ℓ_2:

$$C^1[\varphi_{01}] \neq 0.$$

The function:

$$\varphi_{01} - C^1[\varphi_{01}] h_2$$

is single-valued on S_1.

Let us solve in Ω the problem:

$$\Delta\varphi'=0,$$
$$\varphi'|_{S_1}=\varphi_{01}-C^1[\varphi_{01}]h_2,$$
$$\varphi'|_{S_2}=\varphi_{02}-C^1[\varphi_{01}]h_2$$

(the function $h_2|_{S_1}$, and consequently $\varphi'|_{S_1}$ is determined with accuracy down to a constant term, which we shall regard as fixed). This is the ordinary Dirichlet problem with single-valued boundary conditions.

Let us now assume:

$$\varphi=\varphi'+C^1[\varphi_{01}]h_2.$$

It is obvious that:

$$\varphi|_{S_1}=\varphi_{01};\ \varphi|_{S_2}=\varphi_{02},$$

so that:

$$(\operatorname{grad}\varphi)_\tau|_S=-v_{I\tau}|_S.$$

Thus, the vector $\underline{v}_I$ + $\underline{v}_{II}$ is constructed, such that:

$$\operatorname{div}(\mathbf{v}_I+\mathbf{v}_{II})=0;\ \operatorname{rot}(\mathbf{v}_I+\mathbf{v}_{II})=\mathbf{u};\ (\mathbf{v}_I+\mathbf{v}_{II})_\tau|_S=0.$$

If we now assume that:

$$\mathbf{v}_{III}=-C^2[\mathbf{v}_I+\mathbf{v}_{II}]\operatorname{grad}h_1,$$

then:

$$\mathbf{v}=\mathbf{v}_I+\mathbf{v}_{II}+$$

satisfies all the required conditions.

Lemma 4.3. In order for a smooth vector $\underline{w}$ to be capable of representation as $\underline{w}$ = rot$\underline{u}$, it is necessary and sufficient that:

$$\mathbf{w}\in J',$$

and if this is so, then $\underline{u}$ can be chosen from J.

Necessary condition. If $\underline{w}$ = rot$\underline{u}$, then div$\underline{w}$ = 0 and, by the Stokes formula:

$$C^2[\mathbf{w}] = \int_\Sigma w_n d\Sigma = 0.$$

Sufficient condition. Let $\mathbf{w} \in \hat{J}'$. The vector $\underline{u}_I$ is constructed as in lemma 2.2, only instead of a single Neuman problem we must solve two: one for the region lying within S_2, another for the region between S_1 and a certain attached outer boundary on which a homogeneous condition is imposed. The solvability of these problems is owing to the fact that, by consequence of div$\underline{w}$ = 0 and:

$$\int_\Sigma u_n d\Sigma = 0,$$

we have:

$$\int_{S_1} u_n dS = 0 \text{ and } \int_{S_2} u_n dS = 0.$$

The vector $\underline{u}_{II}$ is constructed as in lemma 2.2 by solving the Neuman problem in Ω.

For $\underline{u}_I$ + $\underline{u}_{II}$ we have:

$$\operatorname{div}(\mathbf{u}_I + \mathbf{u}_{II}) = 0,$$

$$\operatorname{rot}(\mathbf{u}_I + \mathbf{u}_{II}) = \mathbf{w},$$
$$(\mathbf{u}_I + \mathbf{u}_{II})_n|_S = 0.$$

Finally, we construct: /30

$$\mathbf{u}_{III} = -\frac{\int_{S'} (\mathbf{u}_I + \mathbf{u}_{II})_n \, dS'}{\int_{S'} (\operatorname{grad} h_2)_n \, dS'} \operatorname{grad} h_2.$$

The vector:

$$\mathbf{u} = \mathbf{u}_{\mathrm{I}} + \mathbf{u}_{\mathrm{II}} + \mathbf{u}_{\mathrm{III}}$$

satisfies all the required conditions.

<u>Lemma 5.3</u>. For vectors $\underline{v}$ such that div$\underline{v}$ = 0:

$$v_\tau|_S = 0; C^2[\mathbf{v}] = 0,$$

the inequality is valid:

$$\|\mathbf{v}\|_{\mathrm{H}\frac{1}{2}} \leqslant C \|\operatorname{rot} \mathbf{v}\|_{L_2}.$$

If we show that:

$$\|\mathbf{v}\|_{L_2} \leqslant C \|\operatorname{rot} \mathbf{v}\|_{L_2}, \tag{37}$$

then by using lemma 3.2 we can obtain the required inequality in exactly the same way as lemma 4.2.

Let:

$$\operatorname{rot} \mathbf{v} = \mathbf{u} \text{ and } \mathbf{v} = \mathbf{v}_{\mathrm{I}} + \mathbf{v}_{\mathrm{II}} + \mathbf{v}_{\mathrm{III}},$$

where $\underline{v}_{\mathrm{I}}$, $\underline{v}_{\mathrm{II}}$, $\underline{v}_{\mathrm{III}}$ are such as in lemma 3.3. The same as in lemma 4.2, it follows from the formula expressing $\underline{v}_{\mathrm{I}}$ in terms of $\underline{u}$ that:

$$\int_\Omega \mathbf{v}_{\mathrm{I}}^2 dx \leqslant C \int_\Omega \mathbf{u}^2 dx; \quad \int_S v_{\mathrm{I}}^2 dS \leqslant C \int_\Omega \mathbf{u}^2 dx. \tag{38}$$

We shall show that:

$$\int_\Omega \mathbf{v}_{\mathrm{II}}^2 dx = \int_\Omega (\operatorname{grad} \varphi)^2 dx \leqslant C \int_\Omega \mathbf{u}^2 dx. \tag{39}$$

For φ' we have:

$$\int_\Omega (\operatorname{grad} \varphi')^2 dx \leqslant C \int_S (\operatorname{grad} \varphi')_\tau^2 dS.$$

It follows from the boundary conditions for ϕ' that:

$$(\mathrm{grad}\,\varphi')_\tau = v_{1\tau} - C^1[\varphi_{01}](\mathrm{grad}\,h_2)_\tau,$$

so that:

$$\int_\Omega (\mathrm{grad}\,\varphi')^2\,dx \leqslant C\left\{\int_S v_{1\tau}^2\,dS + \int_S (\mathrm{grad}\,h_2)_\tau^2\,dS\cdot\{C^1[\varphi_{01}]\}^2\right\}. \tag{40}$$

We shall show that:

$$\{C^1[\varphi_{01}]\}^2 \equiv \left(\int_{l_2} \mathbf{v}_1\cdot d\mathbf{l}\right)^2 \leqslant C\int_\Omega \mathbf{u}^2\,dx. \tag{41}$$

This inequality follows from inequality:

$$|\mathbf{v}_1| \leqslant C\int_\Omega \frac{|\mathbf{u}(y)|}{|x-y|^2}\,dy$$

(cf. lemma 3.3).

At the left of (41) is a contour integral in 3-dimensional space, so that the properties of potential-type integrals cannot be directly exploited. However this integral does not depend on /3
the choice of contour ℓ_2 on S_1 and therefore can be converted to an integral over a two-dimensional manifold. And indeed, if we consider on S_1 the band σ, interwoven from contours ℓ_2 such that one contour passes through each of its points, then:

$$\left(\int_{l_2} \mathbf{v}_1\cdot d\mathbf{l}\right)^2 \leqslant C\int_\sigma \mathbf{v}_1^2\,dS \leqslant C\int_\Omega \mathbf{u}^2\,dx.$$

Thus, inequality (41) is established. In view of this and (38), we obtain by including:

$$\int_S (\mathrm{grad}\,h_2)_\tau^2\,dS$$

in constant C:

$$\int_\Omega (\mathrm{grad}\,\varphi')^2\,dx \leqslant C\int_\Omega \mathbf{u}^2\,dx.$$

Considering that:

$$\mathbf{v}_{II} = \operatorname{grad} \varphi' + C^1[\varphi_{01}] \operatorname{grad} h_2,$$

and including:

$$\int_\Omega (\operatorname{grad} h_2)^2\, dx$$

in the constant C, it is also easy to obtain (39).

Thus it is demonstrated that:

$$\| \mathbf{v}_I + \mathbf{v}_{II} \|^2_{L_2} \leqslant C \| \mathbf{u} \|^2_{L_2}.$$

In regard to the vector $\underline{v}_{III}$, it is the projection of $\underline{v}_I + \underline{v}_{II}$ onto U_1, and therefore:

$$\| \mathbf{v}_{III} \|^2_{L_2} \leqslant \| \mathbf{v}_I + \mathbf{v}_{II} \|^2_{L_2}.$$

Thus for:

$$\mathbf{v} = \mathbf{v}_I + \mathbf{v}_{II} + \mathbf{v}_{III}$$

We also have inequality (37) and, consequently, the inequality that figures in the lemma.

Remark. The condition $C^2[\underline{v}] = 0$ is important. For example, the lemma is not true for h_1.

If we dispense with this condition, only the inequality is valid:

$$\| \mathbf{v} \|_{W_2^1} \leqslant C (\| \operatorname{rot} \mathbf{v} \|_{L_2} + \| \mathbf{v} \|_{L_2}).$$

Lemma 6.3. For vectors $\mathbf{u} \in \bar{J}$, i.e. those that satisfy the conditions:

$$\operatorname{div} \mathbf{u} = 0; \; u_n|_S = 0; \; \int u_n dS' = 0,$$

the inequality is valid:

$$\|\mathbf{u}\|_{W_2^1} \leqslant C\|\mathrm{rot}\,\mathbf{u}\|_{L_2}.$$

The proof is exactly the same as for a singly-connected region.

Remark. The condition:

$$\int_{S'} u_n dS' = 0$$

is important: the lemma is untrue for $\mathrm{grad}h_2$.

But without this condition the inequality remains valid:

$$\|\mathbf{u}\|_{W_2^1} \leqslant C\left(\|\mathrm{rot}\,\mathbf{u}\|_{L_2} + \|\mathbf{u}\|_{L_2}\right).$$

Similar to Chapter II, theorem 2.3 applies.

Theorem 2.3. Smooth solenoidal vectors $\underline{v}$ with:

$$\mathbf{v}\, c\, v_{\tau}|_S = 0 \text{ and } C^i[\mathbf{v}] = 0.$$

are dense in J'.

If we lift the latter condition, the vectors $\underline{v}$ are dense in J. Analogues of theorems 3.2 and 4.2 of Chapter II follow from the above-given lemmas.

Theorem 3.3. Any vector $\mathbf{u} \in J$ can be represented as /3
$\underline{u}$ = rot$\underline{v}$, where:

$$\mathbf{v} \in W_2^1(\Omega);\ \mathrm{div}\,\mathbf{v} = 0;\ v_{\tau}|_S = 0;\ C^i[\mathbf{v}] = 0,$$

while:

$$\|\mathbf{v}\|_{W_2^1} \leqslant C\|\mathbf{u}\|_{L_2}.$$

In other words, the operator rot establishes a one-to-one correspondence between the lineal of vectors $\underline{v}$ with the indicated properties, dense in J', and all of $\overset{\circ}{J}$.

Theorem 4.3. Any vector $\mathbf{w} \in J'$ can be represented as $\underline{w}$ = rot$\underline{u}$, where:

$$\mathbf{u} \in W_2^1(\Omega);\ \operatorname{div} \mathbf{u} = 0;\ u_n|_S = 0;\ \int_{S'} u_n dS' = 0,$$

while:

$$\|\mathbf{u}\|_{W_2^1} \leqslant C \|\mathbf{w}\|_{L_2}.$$

In other words, the operator rot establishes a one-to-one correspondence between the lineal of these vectors $\underline{u}$, dense in $\overset{\circ}{J}$, and all of J'.

Remark. Inequalities similar to those given in the remark following theorem 4.2 can also be proven for a multiply-connected region.

Theorem 5.3. For the operators of projection onto all the subspaces figuring in theorem 1.3, statements similar to those formulated in theorem 6.2 are valid.

As in the case of a singly-connected region it is sufficient to establish the inequality:

$$\|P\mathbf{u}\|_{W_2^n} \leqslant C \|\mathbf{u}\|_{W_2^n}, \tag{42}$$

where P are projectors onto the particular subspaces and $\underline{u}$ are sufficiently smooth vectors.

To establish the necessary estimate we shall employ the constructions of operators P for smooth $\underline{u}$ given in theorem 1.3.

1. The operators P_J and $P_{\overset{\circ}{G}}$. In this case (42) is obtained in exactly the same way as for a singly-connected region.

2. The operators P_{U_1} and $P_{J'}$. Since, by virtue of resolutions (31)-(34):

$$P_{U_1}\mathbf{u}=P_{U_1}P_J\mathbf{u} \text{ and } P_{J'}\mathbf{u}=P_{J'}P_J\mathbf{u}=(I-P_{U_1})P_J\mathbf{u}$$

and, as follows from the proof of theorem 1.3, for a smooth

$$\mathbf{u}: P_J\mathbf{u}\in \tilde{J},$$

it is sufficient to prove inequality (42) for P_{U_1} and a vector of $\tilde{J}$.

But in this case (cf. item 3 of theorem 1.3):

$$P_{J_1}\mathbf{u}=C^2[\mathbf{u}]\operatorname{grad}h_1,$$

so that:

$$\|P_{U_1}\mathbf{u}\|^2_{W_2^n}\leqslant\|\operatorname{grad}h_1\|^2_{W_2^n}\{C^2[\mathbf{u}]\}^2=\|\operatorname{grad}h_1\|^2_{W_2^n}\left(\int_\Sigma u_n d\Sigma\right)^2\leqslant C\|\mathbf{u}\|^2_{W_2^1(\Omega)}$$

(the latter estimate comes from the nesting theorem of S. L. Sobolev), and this is even more forceful than required.

3. Similar to item 2 it is sufficient to examine the case when $\mathbf{u}\in J'$ and there is only one operator $P_{J'}$.

We shall use the construction of this operator in item 2 of theorem 1.3. The estimate:

$$\|\mathbf{v}\|_{W_2^n}\leqslant C\|\mathbf{u}\|_{W_2^n} \tag{43}$$

is done the same as for a singly-connected region.

The required inequality will be proved if we establish the estimate:

$$C^1[h']\,\mathrm{grad}\,h_2\|^2_{W_2^n} \leqslant C\|\mathbf{u}\|_{W_2^n}$$

or, including the standard vector gradh$_2$ in constant C: /33

$$\{C^1[h']\}^2 \leqslant C\|\mathbf{u}\|^2_{W_2^n}.$$

$C^1[h']$ is the integral of gradh' over a unidimensional manifold ℓ, but does not depend on ℓ, and therefore is evaluated (similar to the fashion of lemma 5.3) in terms of an integral over the two-dimensional manifold σ, while the latter by the nesting theorem of S. L. Sobolev is evaluated in terms of $\|\mathrm{grad}\,h'\|_{W_2^1}$:

$$\{C^1[h']\}^2 \leqslant C\int_\sigma (\mathrm{grad}\,h')^2\,dS \leqslant C\|\mathrm{grad}\,h'\|^2_{W_2^1}.$$

Since:

$$\mathrm{grad}\ h' = \mathbf{u} - \mathbf{v},$$

and estimate (43) is obtained for v, we finally have:

$$\{C^1[h']\}^2 \leqslant C\|\mathbf{u}\|^2_{W_2^1},$$

and this is even more forceful than required.

4. The operators P_{U_2} and $P_{\mathring{J}}$. Similar to the preceding, it is sufficient to examine the case when:

$$\mathbf{u} \in \mathring{J}'$$

and the operator P_{l_1} . Using the expression for $P_{U_2}\underline{u}$ from item 1 of theorem 1.3 we have $\|P_{l_1}\mathbf{u}\|^2_{W_2^n}$ = (constant incl. h$_1$)^2 x

$$\left(\int_{S'} u_n\,dS\right)^2 \leqslant C\|\mathbf{u}\|^2_{W_2^1},$$

and this is even more forceful than required.

In conclusion we formulate the result with respect to a region of general form. This is found quite analogous to the case of the above-considered region.

Let Ω be a region with "n" contours of type:

$$l: l_1, l_2, \ldots, l_n,$$

the boundary of which consists of m surfaces S_1, S_2, ..., S_m. Then, for this, all the statements of the present section are valid, however in this case U_1 is a finite-dimensional subspace of harmonic vectors of type:

$$\alpha_1 \operatorname{grad} h_1 + \alpha_2 \operatorname{grad} h_2 + \ldots + \alpha_m \operatorname{grad} h_m,$$

where h_k are harmonic functions smooth in $\bar{\Omega}$, satisfying boundary conditions:

$$h_k|_{s_i} = \delta_{ik},$$

U_2 is a finite-dimensional subspace of harmonic vectors of type:

$$\beta_1 \operatorname{grad} h^{(1)} + \beta_2 \operatorname{grad} h^{(2)} + \ldots + \beta_n \operatorname{grad} h^{(n)},$$

where $h^{(k)}$ are multi-valued harmonic functions that satisfy conditions:

$$C^{l_i}[h^{(k)}] = \delta_{i,k}, \quad \frac{\partial h^{(k)}}{\partial n}\bigg|_{S_1+\ldots+S_m} = 0.$$

Here:

$$C^{l_i}[h^{(k)}]$$

is a cyclical constant of $h^{(k)}$ in making the circuit of ℓ_i. But in the definitions of $\tilde{J}'$ and $\tilde{J}$ it is necessary to consider surfaces of the type Σ and S', through which the vector fluxes should vanish, as being of all kinds for the given region Ω. It should be noted that the dimensions of U_1 and U_2 in general are smaller than "m" and "n" (e.g. for

the region in the figure), for not all the gradh$_k$ and gradh$^{(k)}$ are linearly-independent.

§2. An Unbounded Region

For simplicity let us examine a region Ω lying outside the bounded surface S, homeomorphic to a sphere.

Several of the changes in the proofs of the theorems, associated with the fact that Ω has as boundary a sphere of infinitely large radius, are already clear in this case.

As concerns the case of a more complicated unbounded region, if all the boundaries (except the infinitely large sphere) are finite and there is a finite number of them, then the region introduces nothing substantially new and merely requires a simple addition of considerations from Chapter III. /34

Subspaces $\overset{\circ}{G}$, $\overset{\circ}{J}$, etc. can be introduced exactly the same as §1 of Chapter II; but the only requirement is that the vectors contained in the lineals $\tilde{G}$, $\tilde{J}$ etc. diminish at infinity as $\frac{1}{|x|^2}$, while their first derivatives as $\frac{1}{|x|^3}$.

Theorem 1.2 concerning the resolution retains its force and is similarly proved. All that should be done from the very start is to take $\underline{u}$ as finite at infinity, and impose an additional condition of diminution of the function at infinity as $\frac{1}{|x|}$, in boundary-value problems for the operator Δ^1.

It is immediately apparent that the vectors $\underline{u}_1$, $\underline{u}_2$, $\underline{u}_3$ turn out to be diminishing at infinity, as required in the definitions of the subspaces.

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[1]With respect to these for an unbounded region cf., e.g., [7].

The theorem of the properties of projectors into the subspaces will also apply.

Theorem 6.3. If a vector $\mathbf{u} \in W_2^n$, its projections into subspaces $\overset{\circ}{G}$, $\overset{\circ}{J}$, G, JU also belong to W_2^n, and for each such projection Pu the inequality is valid:

$$\|P\mathbf{u}\|_{W_2^n} \leqslant C\|\mathbf{u}\|_{W_2^n}.$$

We shall point out the main avenue of proving this theorem. Estimate (27) is obtained as for a bounded region. In obtaining estimate (28) we should further consider that $\tilde{\phi}$ (using a cut function) can be made finite at infinity, while ϕ is such that:

$$D^n \varphi|_\infty \sim \frac{1}{|x|^{n+1}}.$$

Then in the evaluation of grad$(\phi-\tilde{\phi})$ we may again refer to the estimates of O. A. Ladyzhenskaya [14], which remain valid in view of the nature of decrease in $\phi-\tilde{\phi}$ at infinity.

Let us now turn to the proof of estimate (29). As before, we may regard $\tilde{\psi}$ as finite, while ψ is such that:

$$D^n \psi|_\infty \sim \frac{1}{|x|^{n+1}},$$

then from [14] we can obtain the estimate:

$$\int_\Omega \sum_{k=2}^{n} [D^k(\psi-\tilde{\psi})]^2\,dx \leqslant C(\|\mathbf{u}_3^1\|_{W_2^n}^2 + \|\text{grad}\,(\psi-\tilde{\psi})\|_{L_2}^2). \tag{44}$$

For $\tilde{\psi}$, as in Chapter II, the estimate is valid:

$$\|\text{grad}\,\tilde{\psi}\|_{L_2}^2 \leqslant C\|\mathbf{u}_3^1\|_{W_2^n}^2.$$

We shall prove that:

$$\|\text{grad}\,\psi\|_{L_2}^2 \leqslant C\|\mathbf{u}_3^1\|_{W_2^1}^2. \tag{45}$$

Then, inserting this inequality into (44) and taking heed of (27), we obtain (29). /35

ψ is the solution of the problem:

$$\Delta\psi=0,$$
$$\frac{\partial\psi}{\partial n}\Big|_S=-u^1_{3n}|_S,$$
$$\psi|_\infty\sim\frac{1}{|x|}.$$

As is known, this can be represented as the potential of a simple layer:

$$\psi=\int_S\frac{\mu(y)}{|x-y|}dS_y,$$

where $\mu(y)$ is the solution of the integral equation:

$$u^1_{3n}=-2\pi\mu(y)+\int_S\frac{\mu(z)\cos(n_y,z-y)}{|z-y|}dS_z.$$

As is known, in this case we are not on the spectrum and therefore:

$$\|\mu\|_{L_2(S)}\leqslant C\|u^1_{3n}\|_{L_2(S)}.$$

We have:

$$\int_\Omega(\operatorname{grad}\psi)^2dx=\int_S\psi u^1_{3n}dS\leqslant\left\{\int_S(u^1_{3n})^2dS\right\}^{1/2}\left\{\int_S\psi^2dS\right\}^{1/2}\leqslant$$
$$\leqslant C\left\{\int_S(u^1_{3n})^2dS\right\}^{1/2}\left\{\int_S\mu^2dS\right\}^{1/2}\leqslant C\int_S(u^1_{3n})^2dS\leqslant C\|\mathbf{u}^1_3\|^2_{W^1_2}.$$

Thus, theorem 6.3 is entirely proved.

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